

THE UNDECIDABILITY OF THE DEFINABILITY OF PRINCIPAL SUBCONGRUENCES

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ABSTRACT. For each Turing machine \mathcal{T} , we construct an algebra $\mathbb{A}'(\mathcal{T})$ such that the variety generated by $\mathbb{A}'(\mathcal{T})$ has definable principal subcongruences if and only if \mathcal{T} halts, thus proving that the property of having definable principal subcongruences is undecidable. Using this, we present another proof that A. Tarski's finite basis problem is undecidable.

1. INTRODUCTION

Given a variety \mathcal{V} , the *residual bound* of \mathcal{V} is the least cardinal λ that is strictly larger than the cardinality of every subdirectly irreducible member of \mathcal{V} . In the case where such a λ exists we write $\kappa(\mathcal{V}) = \lambda$, and if no such λ exists, then we write $\kappa(\mathcal{V}) = \infty$. If $\mathcal{V} = \mathcal{V}(\mathbb{A})$ is the variety generated by the algebra \mathbb{A} , we define $\kappa(\mathbb{A}) = \kappa(\mathcal{V}(\mathbb{A}))$. It was conjectured that $\kappa(\mathbb{A}) \geq \omega$ implies $\kappa(\mathbb{A}) = \infty$ for finite \mathbb{A} . This is stated in Hobby and McKenzie [4] and is known as the RS-conjecture. McKenzie [7] disproves this conjecture by exhibiting an algebra with residual bound precisely ω . McKenzie then uses this algebra as a basis for his groundbreaking paper [6], in which to each Turing machine \mathcal{T} an algebra $\mathbb{A}(\mathcal{T})$ is associated such that $\kappa(\mathbb{A}(\mathcal{T})) < \omega$ if and only if \mathcal{T} halts, thus proving that the property of is undecidable.

Closely related to the problem of algorithmically determining whether $\kappa(\mathbb{A}) < \omega$ is A. Tarski's finite basis problem. An algebra \mathbb{A} is said to be *finitely based* if the set of identities which are true in \mathbb{A} can be derived from a finite subset of them. Tarski's problem is the question: is there an algorithm that takes as input a finite algebra and determines whether it is finitely based? McKenzie [8] used a construction similar to $\mathbb{A}(\mathcal{T})$ to provide a negative answer to this question, and Willard [9] showed that in fact the original $\mathbb{A}(\mathcal{T})$ is finitely based if and only if \mathcal{T} halts.

One approach to proving that a $\mathcal{V}(\mathbb{A})$ is finitely axiomatizable is to first show that $\kappa(\mathbb{A}) < \omega$, and then to show that $\mathcal{V}(\mathbb{A})$ has a property called *definable principal congruences (DPC)*. These two properties are sufficient to imply that $\mathcal{V}(\mathbb{A})$ is finitely axiomatizable. A first-order formula $\psi(w, x, y, z)$ is said to be a *congruence formula* if for each algebra in the variety, $\psi(w, x, y, z)$ implies $(w, x) \in \text{Cg}(y, z)$ (i.e. that w and x are related by the congruence generated by the pair (y, z)). A class \mathcal{C} of algebras of the same type is said to have definable principal congruences if there is a congruence formula ψ such that for every $\mathbb{B} \in \mathcal{C}$ and every $a, b \in B$ the formula $\psi(-, -, a, b)$ defines $\text{Con}^{\mathbb{B}}(a, b)$. McKenzie [5] shows that if a variety \mathcal{V} has definable principal congruences and $\kappa(\mathcal{V}) < \omega$ then \mathcal{V} is finitely based.

Baker and Wang [2] generalize DPC by saying that a class of algebras \mathcal{C} (all of the same type) has *definable principal subcongruences (DPSC)* if there are congruence

formulas Γ and ψ such that for all $\mathbb{A} \in \mathcal{C}$ and all $a, b \in \mathbb{A}$, if $a \neq b$ then there exist $c, d \in \mathbb{A}$ such that $c \neq d$, $\mathbb{A} \models \Gamma(c, d, a, b)$, and $\psi(-, -, c, d)$ defines $\text{Cg}^{\mathbb{A}}(c, d)$. It is convenient to observe that if the type of \mathcal{C} is finite, then there is a first order formula, $\Pi_{\psi}(y, z)$ which holds in \mathbb{A} if and only if $\psi(-, -, y, z)$ defines $\text{Cg}^{\mathbb{A}}(y, z)$. In symbols, a class \mathcal{C} of algebras of the same finite type has DPSC if there are congruence formulas $\Gamma(w, x, y, z)$ and $\psi(w, x, y, z)$ such that for all $\mathbb{A} \in \mathcal{C}$,

$$\mathbb{A} \models \forall a, b [a \neq b \rightarrow \exists c, d [c \neq d \wedge \Gamma(c, d, a, b) \wedge \Pi_{\psi}(c, d)]] .$$

Baker and Wang [2] use the result that congruence distributive varieties have definable principal subcongruences to give a new proof of K. Baker’s Theorem [1]: if \mathbb{A} is a finite algebra of finite type and $\mathcal{V}(\mathbb{A})$ is congruence distributive, then \mathbb{A} is finitely based. Willard [10] extends Baker’s theorem by showing that if the variety has finite type, is residually finite, and congruence meet-semidistributive ($\mathcal{V}(\mathbb{A}(\mathcal{T}))$ has these features if \mathcal{T} halts), then the variety is finitely based. Since $\mathcal{V}(\mathbb{A}(\mathcal{T}))$ is finitely axiomatizable if and only if \mathcal{T} halts, and finitely axiomatizability is so closely related to DPC and DPSC, it is natural to consider whether the failure of finite axiomatizability when \mathcal{T} does not halt is related to a failure of DPC or DPSC in $\mathcal{V}(\mathbb{A}(\mathcal{T}))$.

The main result of this paper is to construct an algebra $\mathbb{A}'(\mathcal{T})$ based on McKenzie’s $\mathbb{A}(\mathcal{T})$ and to show that $\mathbb{A}'(\mathcal{T})$ has definable principal subcongruences if and only if \mathcal{T} halts. Since the halting problem is undecidable, this proves that the property of having DPSC is undecidable (i.e. there is no algorithm taking a finite algebra as input and outputting whether or not the variety generated by that algebra has DPSC). The proof of this involves many cases, an exploration of “ $\mathbb{A}'(\mathcal{T})$ -arithmetic”, and a fine analysis of the polynomials of $\mathbb{A}'(\mathcal{T})$. We begin in Section 2 with a description of $\mathbb{A}(\mathcal{T})$ and $\mathbb{A}'(\mathcal{T})$. Section 3 details the modifications to McKenzie’s original argument that are necessary to show that $\kappa(\mathbb{A}'(\mathcal{T})) < \omega$ if and only if \mathcal{T} halts. The proof that DPSC is undecidable broken into two cases: if \mathcal{T} halts, and if \mathcal{T} does not halt. The case where \mathcal{T} halts is addressed in Section 4, and is quite complicated. The algebra $\mathbb{A}'(\mathcal{T})$ is the same as McKenzie’s $\mathbb{A}(\mathcal{T})$, but contains an additional operation. This operation makes certain $\mathbb{A}'(\mathcal{T})$ -arithmetical simplifications possible, and is used in key parts of the argument in Section 4. The case where \mathcal{T} does not halt is addressed in Section 5, and a short negative answer to Tarski’s problem using the undecidability of definable principal subcongruences is given in this section as well.

The results in this paper originated with the examination of properties of $\mathbb{A}(\mathcal{T})$. Finite axiomatizability is closely related to the properties of definable principal congruences and definable principal subcongruences, and there was a natural question of whether McKenzie’s answer to Tarski’s finite basis problem was the consequence of a more primitive result concerning either DPC or DPSC. Although it is true that this is the case for $\mathbb{A}'(\mathcal{T})$, it is not clear if it is for the original $\mathbb{A}(\mathcal{T})$. The methods used to prove the undecidability of definable principal subcongruences do not appear to be amenable to proving the undecidability of definable principal congruences, but the overall structure of the argument and the fine analysis of polynomials in $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ may provide a foundation for proving the undecidability of DPC as well.

2. MODIFYING MCKENZIE'S $\mathbb{A}(\mathcal{T})$

We define a *Turing machine* \mathcal{T} to be a finite list of 5-tuples (s, r, w, d, t) , called the *instructions* of the machine, and interpreted as “if in state s and reading r , then write w , move d , and enter state t .” The set of states is finite, $r, w \in \{0, 1\}$, and $d \in \{L, R\}$. A Turing machine takes as input an infinite bidirectional tape $\tau : \mathbb{Z} \rightarrow \{0, 1\}$ which has finite support (i.e. $\tau^{-1}(\{1\})$ is finite). If \mathcal{T} stops computation on some input, then \mathcal{T} is said to have *halted* on that input. Since the tape contains only finitely many nonzero entries, it is possible to encode the tape into the Turing machine. For this reason, we say that the Turing machine halts (without specifying the input) if it halts on the empty tape $\tau(x) = 0$. We will enumerate the states of \mathcal{T} as $\{\mu_0, \dots, \mu_n\}$, where μ_1 is the initial (starting) state, and μ_0 is the halting state (which might not ever be reached). \mathcal{T} has no instruction of the form (μ_0, r, w, d, t) .

Given a Turing machine \mathcal{T} with states $\{\mu_0, \dots, \mu_n\}$, we associate to \mathcal{T} an algebra $\mathbb{A}'(\mathcal{T})$. We will now describe the algebra $\mathbb{A}'(\mathcal{T})$. Let

$$U = \{1, 2, H\}, \quad W = \{C, D, \partial C, \partial D\}, \quad A = \{0\} \cup U \cup W,$$

$$V_{ir}^s = \{C_{ir}^s, D_{ir}^s, M_i^r, \partial C_{ir}^s, \partial D_{ir}^s, \partial M_i^r\} \text{ for } 0 \leq i \leq n \text{ and } \{r, s\} \subseteq \{0, 1\},$$

$$V_{ir} = V_{ir}^0 \cup V_{ir}^1, \quad V_i = V_{i0} \cup V_{i1}, \quad V = \bigcup \{V_i \mid 0 \leq i \leq n\}.$$

The underlying set of $\mathbb{A}'(\mathcal{T})$ is $A'(\mathcal{T}) = A \cup V$. The “ ∂ ” is taken to be a permutation of order 2 with domain $V \cup W$ (e.g. $\partial \partial C = C$), and is referred to as “bar”. It should be mentioned that ∂ is *not* an operation of $\mathbb{A}'(\mathcal{T})$. We now describe the fundamental operations of $\mathbb{A}'(\mathcal{T})$. The algebra $\mathbb{A}'(\mathcal{T})$ is a height 1 meet semilattice (i.e. it is “flat”) with bottom element 0:

$$x \wedge y = \begin{cases} x & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases} \quad \begin{array}{c} x_1 \quad x_2 \quad \dots \\ \quad \diagdown \quad | \quad \diagup \\ \quad \quad 0 \end{array}$$

There is a binary nonassociative “multiplication”, defined by

$$\begin{aligned} 2 \cdot D &= H \cdot C = D, & 1 \cdot C &= C, \\ 2 \cdot \partial D &= H \cdot \partial C = \partial D, & 1 \cdot \partial C &= \partial C, \end{aligned}$$

and $x \cdot y = 0$ otherwise. The next operations play the role of controlling the production of large SI's (those SI's which do not belong to $\mathbf{HS}(\mathbb{A}'(\mathcal{T}))$) in McKenzie's original argument. Such SI's are fully described in Section 3. Define

$$\begin{aligned} J(x, y, z) &= (x \wedge \partial y \wedge z) \vee (x \wedge y) = \begin{cases} x & \text{if } x = y, \\ x \wedge z & \text{if } x = \partial y \in V \cup W, \\ 0 & \text{otherwise,} \end{cases} \\ J'(x, y, z) &= (x \wedge y \wedge z) \vee (x \wedge \partial y) = \begin{cases} x \wedge z & \text{if } x = y, \\ x & \text{if } x = \partial y \in V \cup W, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

and

$$K(x, y, z) = (\partial x \wedge y) \vee (\partial x \wedge \partial y \wedge z) \vee (x \wedge y \wedge z) = \begin{cases} y & \text{if } x = \partial y \in V \cup W, \\ z & \text{if } x = y = \partial z \in V \cup W, \\ x \wedge y \wedge z & \text{otherwise.} \end{cases}$$

(In expressions like $x \wedge \partial y \wedge z$, if y does not lie in the domain of ∂ , then we interpret ∂y as 0). The J and J' operations force a certain kind of nice structure on the SI's of the variety, and the K operation allows us to simplify polynomials of certain forms on the small SI's (this is necessary for to show that $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ has definable principal subcongruences if \mathcal{T} halts). Define

$$\begin{aligned} S_0(u, x, y, z) &= \begin{cases} (x \wedge y) \vee (x \wedge z) & \text{if } u \in V_0, \\ 0 & \text{otherwise,} \end{cases} \\ S_1(u, x, y, z) &= \begin{cases} (x \wedge y) \vee (x \wedge z) & \text{if } u \in \{1, 2\}, \\ 0 & \text{otherwise,} \end{cases} \\ S_2(u, v, x, y, z) &= \begin{cases} (x \wedge y) \vee (x \wedge z) & \text{if } u = \partial v \in V \cup W, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The multiplication operation $y \cdot x$ is, in general, not injective. The next operation will allow us to produce “barred” elements (i.e. produce ∂x from x) in cases when the multiplication is not injective. Let

$$T(w, x, y, z) = \begin{cases} w \cdot x & \text{if } w \cdot x = y \cdot z \text{ and } (w, x) = (y, z), \\ \partial(w \cdot x) & \text{if } w \cdot x = y \cdot z \neq 0 \text{ and } (w, x) \neq (y, z), \\ 0 & \text{otherwise.} \end{cases}$$

Next, we define operations that emulate the computation of the Turing machine. First, we define an operation that when applied to certain elements of $\mathbb{A}'(\mathcal{T})^{\mathbb{Z}}$ will produce something that represents a “blank tape”:

$$I(x) = \begin{cases} C_{10}^0 & \text{if } x = 1, \\ M_1^0 & \text{if } x = H, \\ D_{10}^0 & \text{if } x = 2, \\ 0 & \text{otherwise.} \end{cases}$$

For each instruction of \mathcal{T} of the form (μ_i, r, s, L, μ_j) and each $t \in \{0, 1\}$ define an operation

$$L_{irt}(x, y, u) = \begin{cases} C_{jt}^{s'} & \text{if } x = y = 1 \text{ and } u = C_{ir}^{s'} \text{ for some } s', \\ M_j^t & \text{if } x = H, y = 1, \text{ and } u = C_{ir}^t, \\ D_{jt}^s & \text{if } x = 2, y = H, \text{ and } u = M_i^r, \\ D_{jt}^{s'} & \text{if } x = y = 2 \text{ and } u = D_{ir}^{s'} \text{ for some } s', \\ \partial v & \text{if } u \in V \text{ and } L_{irt}(x, y, \partial u) = v \in V \text{ by the above lines,} \\ 0 & \text{otherwise.} \end{cases}$$

Let \mathcal{L} be the set of all such operations. Similarly, for each instruction of \mathcal{T} of the form (μ_i, r, s, R, μ_j) and each $t \in \{0, 1\}$ define an operation

$$R_{irt}(x, y, u) = \begin{cases} C_{jt}^{s'} & \text{if } x = y = 1 \text{ and } u = C_{ir}^{s'} \text{ for some } s', \\ C_{jt}^s & \text{if } x = H, y = 1, \text{ and } u = M_i^r, \\ M_j^t & \text{if } x = 2, y = H, \text{ and } u = D_{ir}^t, \\ D_{jt}^{s'} & \text{if } x = y = 2 \text{ and } u = D_{ir}^{s'} \text{ for some } s', \\ \partial v & \text{if } u \in V \text{ and } R_{irt}(x, y, \partial u) = v \in V \text{ by the above lines,} \\ 0 & \text{otherwise.} \end{cases}$$

Let \mathcal{R} be the set of all such operations. When applied to certain elements from $\mathbb{A}'(\mathcal{T})^{\mathbb{Z}}$, these operations simulate the computation of the Turing machine \mathcal{T} on different inputs. Certain elements of $\{1, 2, H\}^{\mathbb{Z}}$ serve to track the position of the Turing machine's head when operations from $\mathcal{L} \cup \mathcal{R}$ are applied to elements of $\mathbb{A}'(\mathcal{T})^{\mathbb{Z}}$ that encode the contents of the tape. For this reason, we define a binary relation \prec on $\{1, 2, H\}$ by $x \prec y$ if and only if $x = y = 2$, or $x = 2$ and $y = H$, or $x = y = 1$. For $F \in \mathcal{L} \cup \mathcal{R}$ note that $F(x, y, z) = 0$ except when $x \prec y$. As with multiplication, operations from $\mathcal{L} \cup \mathcal{R}$ are not injective. The next two operations allow us to produce barred elements in some cases when an operation from $\mathcal{L} \cup \mathcal{R}$ is not injective. Define two operations for each $F \in \mathcal{L} \cup \mathcal{R}$,

$$U_F^1(x, y, z, u) = \begin{cases} \partial F(x, y, u) & \text{if } x \prec z, y \neq z, F(x, y, u) \neq 0, \\ F(x, y, u) & \text{if } x \prec z, y = z, F(x, y, u) \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$U_F^0(x, y, z, u) = \begin{cases} \partial F(y, z, u) & \text{if } x \prec z, x \neq y, F(y, z, u) \neq 0, \\ F(y, z, u) & \text{if } x \prec z, x = y, F(y, z, u) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

The operations on $\mathbb{A}'(\mathcal{T})$ are

$$\{0, \wedge, (\cdot), J, J', K, S_0, S_1, S_2, T, I\} \cup \mathcal{L} \cup \mathcal{R} \cup \{U_F^1, U_F^2 \mid F \in \mathcal{L} \cup \mathcal{R}\}.$$

The only difference between this algebra and the algebra McKenzie used in [6] is the addition of the K operation. McKenzie used his $\mathbb{A}(\mathcal{T})$ (without the K operation) to prove the following theorem.

Theorem 2.1 (McKenzie [6]). $\kappa(\mathbb{A}(\mathcal{T})) < \omega$ if and only if \mathcal{T} halts.

The fact that $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ has only finitely many subdirectly irreducible algebras, all finite if \mathcal{T} halts is needed to prove that it has definable principal subcongruences. Since we have modified the algebra that this theorem refers to, we must show that this theorem (as well as many other important properties of $\mathbb{A}(\mathcal{T})$) still holds.

3. MODIFYING MCKENZIE'S ARGUMENT

McKenzie's argument is quite detailed and long, and fortunately only needs to be added to – not changed. In this section we will detail the specific additions that are necessary for the arguments appearing in papers [7] and [6] to still hold. We begin by determining a formula for K on the large subdirectly irreducible algebras.

Lemma 3.1. *Let $\mathbb{B} \in \mathcal{V}(\mathbb{A}'(\mathcal{T}))$. In \mathbb{B} , if $S_2(u, v, x, y, z) \approx 0$ then $J(x, y, z) \approx x \wedge y$ and $J'(x, y, z) \approx K(x, y, z) \approx x \wedge y \wedge z$.*

Proof. Pick an index set L , $\mathbb{C} \leq \mathbb{A}'(\mathcal{T})^L$, and $\theta \in \text{Con}(\mathbb{C})$ such that $\mathbb{B} = \mathbb{C}/\theta$. Suppose that $S_2(u, v, x, y, z) \approx 0$ in $\mathbb{B} = \mathbb{C}/\theta$. If $u, v \in C$ are such that $u(l) = \partial v(l)$ for some $l \in L$, then $S_2(u, v, u, u, u)(l) = u(l)$ and $S_2(u, v, v, v, v)(l) = v(l)$, so it must be that $(u, 0), (v, 0) \in \theta$. Thus, either C contains no elements u, v such that $u(l) = \partial v(l)$ for some l , or all such elements are related to 0 by θ . It follows that in the definition of the K operation, we have

$$\begin{aligned} K(x, y, z) &= \begin{cases} y & \text{if } x = \partial y \in W \cup V, \\ z & \text{if } x = y = \partial z \in W \cup V, \\ x \wedge y \wedge z & \text{otherwise,} \end{cases} \\ &= \begin{cases} y & \text{if } x = \partial y = 0 \bmod \theta, \\ z & \text{if } x = y = \partial z = 0 \bmod \theta, \\ x \wedge y \wedge z & \text{otherwise,} \end{cases} \\ &= x \wedge y \wedge z \bmod \theta. \end{aligned}$$

Therefore in \mathbb{B} we have $K(x, y, z) = x \wedge y \wedge z$. The argument for the J and J' operations is almost identical. \square

Many additions occur where induction on polynomial complexity is used, and the following lemma is the crux of the additional argumentation in most of these instances.

Lemma 3.2. *Let L be an index set and suppose that $\mathbb{B} \leq \mathbb{A}'(\mathcal{T})^L$ and $C \subseteq B$ are such that*

- (1) *if $c \in C$ then $c(l) \neq 0$ for all $l \in L$ (we will say that c is nowhere 0), and*
- (2) *if $c \in C$ and $a \in B$ are such that $c(l) \in \{a(l), \partial a(l)\}$ for all $l \in L$, then $c = a$.*

If $f_1(x), f_2(x), f_3(x)$ are polynomials of \mathbb{B} such that for all i either $f_i(x)$ is constant or $f_i^{-1}(C) \subseteq C$, then the polynomial $f(x) = K(f_1(x), f_2(x), f_3(x))$ is also either constant or $f^{-1}(C) \subseteq C$.

Proof. Let $f_1(x), f_2(x), f_3(x)$ be as in the statement of the lemma, and let $f(x) = K(f_1(x), f_2(x), f_3(x))$. We will show that $f(x)$ is either constant or $f^{-1}(C) \subseteq C$. Suppose that $a \in B$ and $f(a) \in C$. Since

$$\begin{aligned} f(a) &= K(f_1(a), f_2(a), f_3(a)) \\ &= (\partial f_1(a) \wedge f_2(a)) \vee (\partial f_1(a) \wedge \partial f_2(a) \wedge f_3(a)) \vee (f_1(a) \wedge f_2(a) \wedge f_3(a)), \end{aligned}$$

and $f(a)$ is nowhere 0, for each $l \in L$ either

- $\partial f_1(a)(l) = f_2(a)(l) = f(a)(l)$, or
- $\partial f_1(a)(l) = \partial f_2(a)(l) = f_3(a)(l) = f(a)(l)$, or
- $f_1(a)(l) = f_2(a)(l) = f_3(a)(l) = f(a)(l)$.

Therefore $f(a)(l) \in \{f_1(a)(l), \partial f_1(a)(l)\}$. Since $f(a) \in C$, by hypothesis (2) this implies that $f(a) = f_1(a)$, which is only possible if $f_1(a) = f_2(a) = f_3(a)$. Since this holds for every $a \in B$, we have that $f(x) = f_1(x) = f_2(x) = f_3(x)$. By the hypothesis, for each $i \in \{1, 2, 3\}$, $f_i(x)$ is either constant or $f_i^{-1}(C) \subseteq C$. Since all of the $f_i(x)$ agree and are equal to $f(x)$, this immediately implies that $f(x)$ is either constant or $f^{-1}(C) \subseteq C$. \square

Definition 3.3. Let \mathcal{C} be a class of algebras of the same type whose reduct to $\{0, \wedge\}$ is a meet semilattice. \mathcal{C} is said to be *0-absorbing* if for every fundamental operation $F(x_1, \dots, x_n)$, every $\mathbb{A} \in \mathcal{C}$, and every $a_1, \dots, a_n \in A$,

$$0 \in \{a_1, \dots, a_n\} \quad \text{implies} \quad F(a_1, \dots, a_n) = 0.$$

\mathcal{C} is said to *commute with \wedge* if for every fundamental operation $F(x_1, \dots, x_n)$, $\mathbb{A} \in \mathcal{C}$, and $a_1, b_1, \dots, a_n, b_n \in A$,

$$F(a_1, \dots, a_n) \wedge F(b_1, \dots, b_n) = F(a_1 \wedge b_1, \dots, a_n \wedge b_n).$$

We now enumerate the additions to McKenzie's proofs in papers [7] and [6]. To avoid needlessly long definitions and discussions, the additions will be presented assuming that the reader has the appropriate paper on hand to reference. Overall, we will proceed through the argument in [6], and divert to [7] when the main argument makes reference to it.

- (1) In general, we note that K is monotonic, and if $\mathbb{A} \in \mathcal{V}(\mathbb{A}'(\mathcal{T}))$ is such that $\mathbb{A} \models S_2(u, v, x, y, z) \approx 0$, then $\mathbb{A} \models J'(x, y, z) \approx K(x, y, z) \approx x \wedge y \wedge z$. The first part of this observation is used throughout the proof. For the second part of the observation, [6] Lemma 5.2 (i), Lemma 5.5, and noting that the element p in these lemmas is nowhere 0, implies that in the large SI's $S_i(\overline{u}, x, y, z) \approx 0$ for $i \in \{0, 1, 2\}$. Lemma 3.1 then yields the claimed identities. Since $K(x, y, z) \approx J'(x, y, z)$ in the large SI's, the addition of the K operation does not change the structure of large SI's in $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$.
- (2) In [6] in the proof of Lemma 4.1, elements α_n and β_n of $\mathbb{A}'(\mathcal{T})^{\mathbb{Z}}$ are defined as

$$\alpha_n(k) = \begin{cases} 1 & \text{if } k < n, \\ H & \text{if } k = n, \\ 2 & \text{if } k > n, \end{cases} \quad \beta_n(k) = \begin{cases} C & \text{if } k < n, \\ D & \text{if } k \geq n. \end{cases}$$

Let Γ be the subuniverse of the algebra generated by these elements, Σ the set of all configuration elements generated by the α_n (the set of all nowhere 0 outputs of $\mathcal{L} \cup \mathcal{R} \cup \{I\}$), and Γ_0 the subset of Γ consisting of elements that are 0 at some coordinate. It is necessary to prove that the set

$$\Gamma' = \Gamma_0 \cup \Sigma \cup \{\alpha_n, \beta_n \mid n \in \mathbb{Z}\}$$

is closed under the operation K . By construction, if $u \in \Gamma' \setminus \Gamma_0$, then for each $l \in L$, $u(l)$ cannot be a barred element (e.g. ∂C , ∂D , ∂C_{ir}^s , etc.). From the definition of K , we have that if $a, b, c \in \Gamma' \setminus \Gamma_0$, then $K(a, b, c) = a \wedge b \wedge c \in \Gamma$. The set Γ_0 contains elements that have a value of 0 at some coordinate. Since K is 0-absorbing in its first and second coordinates, $K(\Gamma_0, \Gamma', \Gamma') \subseteq \Gamma_0$. Furthermore, if $a, b \in \Gamma' \setminus \Gamma_0$ and $c \in \Gamma_0$ then since $a(l) \neq \partial b(l)$ for any $l \in \mathbb{Z}$ (Γ' contains no barred elements), we have that $K(a, b, c) = a \wedge b \wedge c$, so in this case $K(a, b, c) \in \Gamma_0$ since $w \in \Gamma_0$. Therefore $K(\Gamma' \setminus \Gamma_0, \Gamma' \setminus \Gamma_0, \Gamma_0) \subseteq \Gamma_0$.

- (3) In [6] in the proof of Lemma 5.3, an inductive argument on the complexity of polynomials is used. In the proof, McKenzie considers an algebra $\mathbb{B} \leq \mathbb{A}'(\mathcal{T})^L$ for some index set L , and $p \in B$ such that p is nowhere 0 and if $u \in B$ is such that if $u(l) = p(l)$ or $u(l) = \partial p(l)$ for all $l \in L$, then $u = p$. The claim to be proved is that if $g(x)$ is a polynomial of \mathbb{B} then $g(x)$

is constant or $g^{-1}(p) \subseteq \{p\}$. The inductive step of this claim is exactly Lemma 3.2 with $C = \{p\}$.

- (4) Prior to the statement of Lemma 5.5 in [6], it is written that the lemma is a restatement of Lemmas 6.7-6.9 of [7]. All of these lemmas go through without modification, except for Lemma 6.8. Lemma 6.8 concerns itself with a subalgebra \mathbb{B} of $\mathbb{A}'(\mathcal{T})^L$ and a congruence θ of \mathbb{B} such that \mathbb{B}/θ is SI. This congruence turns out to be generated by the pair (p, q) , and various properties of p and q are given in these lemmas. For the addition described below, it is only necessary to know that p is nowhere 0. Let

$$B_1 = \{u \in B \mid u = p \text{ or } x_0x_1 \cdots x_n = p \text{ and } u \in \{x_0 \dots, x_n\}\}$$

(the product in $x_0x_1 \cdots x_n$ associates to the right). At the very start of the proof of Lemma 6.8, induction on the complexity of polynomials is used to prove that if $u \in B$ and $f(u) \in B_1$ then $f(x)$ is either constant or $u \in B_1$. Lemma 6.6 in [7] states that B_1 consists of elements that are nowhere 0 and such that if $u \in B_1$ and $v \in B$ are such that $u(l) = v(l)$ or $u(l) = \partial v(l)$ for all $l \in L$, then $u = v$. Taking $C = B_1$ in Lemma 3.2 above, the inductive step of the proof follows immediately.

- (5) In [6] in the proof of Lemma 5.7 part (iii), induction on the complexity of polynomials is used to prove that if $f(x)$ is a polynomial of \mathbb{B} and $f(u) \in B_1$ for some $u \in B$, then $u \in B_1$. This is the same argument that appears in the previous item above.

This completes the changes that are needed to adapt McKenzie's description of large SI algebras in $\mathcal{V}(\mathbb{A}(\mathcal{T}))$ to $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$. We will now give a rather explicit description of exactly what these algebras look like.

Large subdirectly irreducible algebras in $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ come in two types: sequential and machine. Both of these types of algebras model $S_i(\overline{n}, x, y, z) \approx 0$. Sequential algebras are distinguished as additionally modeling the identities $I(x) \approx F(x, y, z) \approx 0$ for all $F \in \mathcal{L} \cup \mathcal{R}$. Machine type algebras are distinguished as modeling the identities $x \cdot y \approx T(w, x, y, z) \approx 0$ instead. We begin by describing the sequential algebras.

We begin the description of the sequential type algebras by describing an algebra $\mathbb{S}_{\mathbb{Z}}$ in which every sequential type algebra is embeddable (but which may not belong to $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$). The algebra $\mathbb{S}_{\mathbb{Z}}$ has underlying set $S_{\mathbb{Z}} = \{0, a_i, b_i \mid i \in \mathbb{Z}\}$ and fundamental operations of $S_{\mathbb{Z}}$ are the same as $\mathbb{A}'(\mathcal{T})$, but are all identically 0 except for \wedge , (\cdot) , T , J , J' , and K . The operation \wedge is defined so that $\langle S_{\mathbb{Z}}; \wedge \rangle$ is a flat meet semilattice with bottom element 0. The operation (\cdot) is defined so that $a_n \cdot b_{n+1} = b_n$, and 0 otherwise. The operations T , J , J' , and K are defined

$$\begin{aligned} J(x, y, z) &= x \wedge y, & J'(x, y, z) &= K(x, y, z) = x \wedge y \wedge z, \\ T(w, x, y, z) &= (w \cdot x) \wedge (y \cdot z). \end{aligned}$$

Define \mathbb{S}_{ω} to be the subalgebra of $\mathbb{S}_{\mathbb{Z}}$ with universe $\{0, a_i, b_i \mid i \geq 0\}$, and define \mathbb{S}_n to be the subalgebra of $\mathbb{S}_{\mathbb{Z}}$ with universe $\{0, a_1, b_1, \dots, a_n, b_n\}$. The algebras \mathbb{S}_{ω} and \mathbb{S}_n are subdirectly irreducible, with monoliths $\text{Cg}(b_0, 0)$. McKenzie [6] and the additions enumerated in above (in particular, item (2)) proves that $\mathbb{S}_{\mathbb{Z}} \in \mathcal{V}(\mathbb{A}'(\mathcal{T}))$ if and only if \mathcal{T} does not halt, and that \mathcal{T} halts if and only if there is some maximum $N \in \mathbb{N}$ such that $\mathbb{S}_N \in \mathcal{V}(\mathbb{A}'(\mathcal{T}))$. $\mathbb{S}_{\mathbb{Z}}$, \mathbb{S}_{ω} , and \mathbb{S}_n for $n \in \mathbb{N}$ are the sequential algebras, but only \mathbb{S}_n and \mathbb{S}_{ω} are subdirectly irreducible.

Next, we restate the description of machine type algebras given by McKenzie [6]. We begin by describing an algebra (possibly not in $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$) that will have a quotient isomorphic to our hypothetical machine type algebra. Let $N \subseteq \mathbb{Z}$ be a nonempty interval and let $\mathcal{Q} = \langle \tau, j, \gamma \rangle$ be any configuration of the Turing machine \mathcal{T} (here τ is the tape function, $j \in N$ is the head position, and γ is the state of the machine). We say that \mathcal{Q} is an initial configuration if τ is the blank tape (the tape consisting of all 0's, written as τ_0 below) and $\gamma = \mu_1$ (the starting state). We say that \mathcal{Q} is a halting configuration if $\gamma = \mu_0$ (the halting state). Let Ω_N denote the set of all configurations $\langle \tau, j, \gamma \rangle$ with $j \in N$. Write $\mathcal{P} \leq_N \mathcal{Q}$ if there is a finite sequence $\mathcal{Q} = \mathcal{Q}_0, \dots, \mathcal{Q}_m = \mathcal{P}$ with $\mathcal{Q}_i \in \Omega_N$ and such that $\mathcal{Q}_{i+1} = \mathcal{T}(\mathcal{Q}_i)$.

Let $\Sigma_N = \{a_n \mid n \in N\}$, and assume that Σ_N , Ω_N , and $\{0\}$ are pairwise disjoint. Let \mathbb{P}_N be the algebra where

- the universe is $P_N = \{0\} \cup \Sigma_N \cup \Omega_N$.
- the operations $(\cdot), S_0, S_1, S_2, T$ are identically 0.
- \wedge makes $\langle P_N, \wedge \rangle$ a flat semilattice.
- $J(x, y, z) = x \wedge y$ and $J'(x, y, z) = K(x, y, z) = x \wedge y \wedge z$.
- $I(a_n) = \langle \tau_0, n, \mu_1 \rangle \in \Omega_N$ and $I(x) = 0$ otherwise (here τ_0 is the tape consisting of all 0's).
- if $F = L_{ir\varepsilon} \in \mathcal{L}$ where $\mu_i r s L \mu_j$ is an instruction of \mathcal{T} and $\mathcal{Q} = \langle \tau, n+1, \mu_i \rangle$ is a configuration in Ω_N , then $F(a_n, a_{n+1}, \mathcal{Q}) = \mathcal{T}(\mathcal{Q})$ provided that $n \in N$, $\mathcal{T}(\mathcal{Q}) \in \Omega_N$, $\tau(n+1) = r$, and $\tau(n) = \varepsilon$. In all other cases $F(x, y, z) = 0$. The case when $F = R_{ir\varepsilon} \in \mathcal{R}$ is defined analogously.
- if $F \in \mathcal{L} \cup \mathcal{R}$ and $n, n+1 \in N$, we have

$$U_F^1(a_n, a_{n+1}, a_{n+1}, x) = F(a_n, a_{n+1}, x) = U_F^2(a_n, a_n, a_{n+1}, x),$$

and $U_F^j(w, x, y, z) = 0$ otherwise.

Next, we describe the congruence of \mathbb{P}_N which we will quotient by. Assume the set $\Phi \subseteq \Omega_N$ and the element $\mathcal{P} \in \Phi$ satisfy the following conditions.

- (1) For all $\mathcal{Q} \in \Phi$ we have $\mathcal{P} \leq_N \mathcal{Q}$.
- (2) If $\mathcal{Q} \in \Phi$ and $\mathcal{P} \leq_N \mathcal{T}(\mathcal{Q})$ then $\mathcal{T}(\mathcal{Q}) \in \Phi$.
- (3) If $\mathcal{Q} \in \Omega_N$ is an initial configuration and $\mathcal{P} \leq_N \mathcal{Q}$ then $\mathcal{Q} \in \Phi$.
- (4) If $\mathcal{Q}, \mathcal{Q}' \in \Omega_N$, \mathcal{Q}' is a halting configuration, and $\mathcal{Q}' \leq_N \mathcal{Q}$ then $\mathcal{Q} \notin \Phi$.
- (5) $|N| > 1$ and for every $n \in N$, there is some $\langle \tau, n, \gamma \rangle \in \Phi$.

Define Γ to be $(\Omega_N \setminus \Phi) \cup \{0\}$ and let $\Theta_{(\Phi)}$ be the congruence of \mathbb{P}_N whose only nontrivial class is Γ . McKenzie gives the following theorem at the end of [6], which with the addition of the arguments above still holds for the modified $\mathbb{A}'(\mathcal{T})$.

Theorem 3.4 (McKenzie [6]). *$\Theta_{(\Phi)}$ is a congruence relation of \mathbb{P}_N and the algebra $\mathbb{P}_N/\Theta_{(\Phi)}$ is a subdirectly irreducible algebra that belongs to $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$. Every large SI in $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ is either embeddable in \mathbb{S}_ω or is isomorphic to $\mathbb{P}_N/\Theta_{(\Phi)}$ for some N and Φ as above.*

The above description of the SI algebras in $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ extends McKenzie's result that $\kappa(\mathbb{A}(\mathcal{T})) < \omega$ if and only if \mathcal{T} halts to $\mathbb{A}'(\mathcal{T})$.

Theorem 3.5. *$\kappa(\mathbb{A}'(\mathcal{T})) < \omega$ if and only if \mathcal{T} halts.*

4. IF \mathcal{T} HALTS

The argument to show that $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ has definable principal subcongruences if \mathcal{T} halts is quite long and intricate, so we will begin by giving an outline of how the various lemmas and theorems tie together.

If \mathbb{B} is an algebra, then by Maltsev's Lemma, $(c, d) \in \text{Cg}^{\mathbb{B}}(a, b)$ if and only if there is a sequence of elements, $c = k_1, k_2, \dots, k_n = d$, terms f_1, \dots, f_{n-1} , and constants $\bar{e} \in B^m$ such that $\{f_i(\bar{e}, a), f_i(\bar{e}, b)\} = \{k_i, k_{i+1}\}$ for all i . A *congruence scheme*, as in [3], is a first-order formula, $\varphi(w, x, y, z)$, that asserts the existence of such elements k_1, \dots, k_n and constants \bar{e} for some fixed sequence of terms. A disjunction of congruence schemes is a *congruence formula*, and every $(c, d) \in \text{Cg}^{\mathbb{B}}(a, b)$ satisfies some congruence scheme. Thus, showing that a principal congruence is definable can be reduced to finding a finite number of schemes that fully describe the congruence, and showing that a variety has definable principal congruences can be reduced to showing that there is a finite number of congruence schemes that fully describe every principal congruence in every algebra in the variety.

Begin with an arbitrary $\mathbb{B} \in \mathcal{V}(\mathbb{A}'(\mathcal{T}))$ with subdirect representation $\mathbb{B} \leq \prod_{l \in L} \mathbb{C}_l$ where each \mathbb{C}_l is subdirectly irreducible. Define $e_i(\bar{n}, x) = S_i(\bar{n}, x, x, x)$, where $\bar{n} = n_1$ if $i \in \{0, 1\}$ and $\bar{n} = (n_1, n_2)$ if $i = 2$. The isomorphism types of the \mathbb{C}_l come in 4 different flavors.

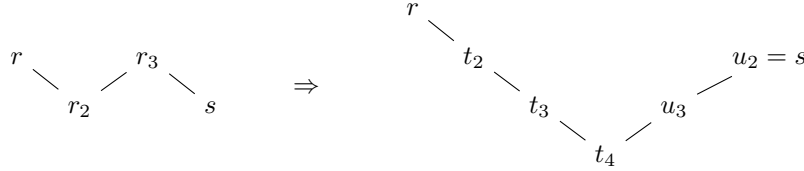
- (1) $\mathbb{C}_l \models \exists \bar{n}[e_i(\bar{n}, x) \approx x]$ for some $i \in \{0, 1, 2\}$. In this case, \mathbb{C}_l is necessarily small (see Lemma 5.2 in [6]), and if $\bar{m} \in B^2 \cup B$ is such that $\bar{m}(l) = \bar{n}$ then $e_i(\bar{m}, \mathbb{B})$ is congruence distributive (see the proof of Lemma 4.2). The class of all algebras of $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ that model $\exists \bar{n}[e_i(\bar{n}, x) \approx x]$ has definable principal subcongruences (see Lemma 4.2).
- (2) \mathbb{C}_l is small and $\mathbb{C}_l \models e_i(\bar{y}, x) \approx 0$ for all $i \in \{0, 1, 2\}$. In this case there are just 3 isomorphism types (see Lemma 4.17).
- (3) \mathbb{C}_l is large and $\mathbb{C}_l \models e_i(\bar{y}, x) \approx 0$ for all $i \in \{0, 1, 2\}$ and $\mathbb{C} \models I(x) \approx F(x, y, z) \approx 0$ for all $F \in \mathcal{L} \cup \mathcal{R}$. In this case, \mathbb{C} is said to be of *sequential type* and has a nice structure based on the (\cdot) operation. SI's of this type are fully described in Section 3.
- (4) \mathbb{C}_l is large and $\mathbb{C} \models e_i(\bar{y}, x) \approx 0$ for all $i \in \{0, 1, 2\}$ and $\mathbb{C} \models x \cdot y \approx T(w, x, y, z) \approx 0$. In this case, \mathbb{C} is said to be of *machine type* and has a nice structure based on the machine operations $\mathcal{L} \cup \mathcal{R}$. SI's of this type are fully described in Section 3.

In order to show that $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ has definable principal subcongruences, we will produce congruence formulas Γ and ψ such that for any $\mathbb{B} \in \mathcal{V}(\mathbb{A}'(\mathcal{T}))$ and any $a', b' \in B$ there is $(c, d) \in \text{Cg}^{\mathbb{B}}(a', b')$ witnessed by $\Gamma(c, d, a', b')$ and such that the relation “ $(x, y) \in \text{Cg}^{\mathbb{B}}(c, d)$ ” is defined by $\psi(x, y, c, d)$. Let $\mathbb{B} \leq \prod_{l \in L} \mathbb{C}_l$ be a subdirect representation of \mathbb{B} by subdirectly irreducible algebras. The way in which (c, d) is produced depends on the isomorphism types of the \mathbb{C}_l such that $a'(l) \neq b'(l)$. Our first step is to assume without loss of generality that $a' \not\leq b'$ and to take $a = a'$ and $b = a' \wedge b'$ so that $b \leq a$. Let $K = \{l \in L \mid a(l) \neq b(l)\}$. The case distinctions are

- (1) There is $l \in K$ such that $\mathbb{C}_l \models \exists \bar{n}[e_i(\bar{n}, x) \approx x]$. These are the SI's described in item (1) above.
- (2) The previous case does not hold, and there is $l \in K$ such that \mathbb{C}_l is of sequential type, and either the operation (\cdot) distinguishes a and b , or

- $(a(l), b(l))$ lies in the monolith of \mathbb{C}_l for all $l \in L$. These are the SI's described in item (3) above.
- (3) The previous cases do not hold, and there is $l \in K$ such that \mathbb{C}_l is of machine type, and either some of the operations of $\mathcal{L} \cup \mathcal{R} \cup \{I\}$ distinguish a and b , or $(a(l), b(l))$ lies in the monolith of \mathbb{C}_l for all $l \in L$. These are the SI's described in item (4) above.
- (4) The previous cases do not hold, in which case it must be that the only $l \in K$ are such that \mathbb{C}_l is one of the three small SI's that satisfy $e_i(\bar{n}, x) \approx 0$ for all $i \in \{0, 1, 2\}$. These are the SI's described in item (2) above.

The goal is to take (a, b) and to uniformly produce $(c, d) \in \text{Cg}^{\mathbb{B}}(a, b)$ such that $\text{Cg}^{\mathbb{B}}(c, d)$ is uniformly definable. In all of the cases, the way in which (c, d) is produced will ensure that $d \leq c$. In order to show that $\text{Cg}^{\mathbb{B}}(c, d)$ is definable, we will need to consider Maltsev chains witnessing arbitrary $(r, s) \in \text{Cg}^{\mathbb{B}}(c, d)$. The first reduction is to apply Maltsev's Lemma to get that if $(r, s) \in \text{Cg}^{\mathbb{B}}(c, d)$, then it is witnessed by a Maltsev chain whose associated polynomials are generated by fundamental translations. The next reduction is to sequentially meet each element of the chain, starting at r , so that it is strictly decreasing, and to do it again but starting at s instead. This produces two strictly decreasing chains and some intermediate element $t \in B$ (the meet of all elements in the chain) such that $t \leq r \wedge s$ and $(r, t), (t, s) \in \text{Cg}^{\mathbb{B}}(c, d)$. Thus, we may restrict our analysis of Maltsev chains to analyzing two chains that are strictly decreasing and share a common endpoint and with associated polynomials all generated by fundamental translations.



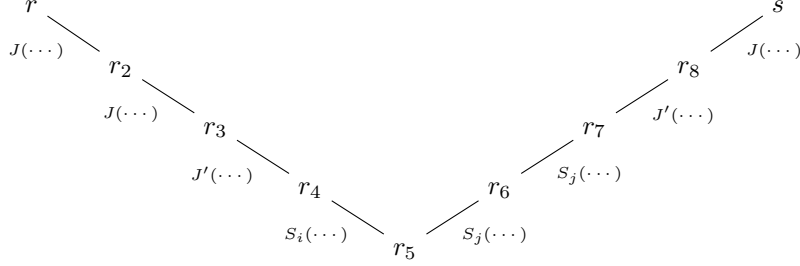
In Case 1, there is some $i \in \{0, 1, 2\}$ and $\bar{n} \in B^2 \cup B$ such that $e_i(\bar{n}, a) \neq e_i(\bar{n}, b)$. Fixing one such \bar{n} and i , let $c' = e_i(\bar{n}, a)$ and $d' = e_i(\bar{n}, b)$. From the observations on the \mathbb{C}_l above, this gives us that $c'(l) = d'(l) = 0$ if $\mathbb{C}_l \not\models e_i(\bar{n}(l), x) \approx x$, and

$$c'(l) = a(l) \geq b(l) = d'(l)$$

if $\mathbb{C}_l \models e_i(\bar{n}(l), x) \approx x$. Lemma 4.2 shows that $\text{Cg}^{e_i(\bar{n}, \mathbb{B})}(c', d')$ has a definable principal subcongruence; call it $\text{Cg}^{e_i(\bar{n}, \mathbb{B})}(c, d)$, and without loss of generality assume that $d \leq c$. The aim of the proof for Case 1 is to show that the congruence $\text{Cg}^{\mathbb{B}}(c, d)$ is definable. If all the polynomials of \mathbb{B} were 0-absorbing, then we would have that $\text{Cg}^{\mathbb{B}}(c, d) = \text{Cg}^{e_i(\bar{n}, \mathbb{B})}(c, d)$, but the fundamental operations J , J' , K , and S_i are not 0-absorbing. The analysis of the polynomials of $\mathbb{A}'(\mathcal{T})$ will therefore focus on quantifying the extent to which these operations are not 0-absorbing, and describing how they interact with the other fundamental operations.

Assume that $(r, s) \in \text{Cg}^{\mathbb{B}}(c, d)$ is witnessed by a strictly decreasing Maltsev chain with associated polynomials generated by fundamental translations. Lemma 4.4 proves that we can assume the associated polynomials have a J , J' , or K operation in their decomposition or are in the range of S_j for some $j \in \{0, 1, 2\}$, and Lemmas 4.5, 4.6, and 4.7 prove that if we allow the length of the chain to grow, we can ensure that the operations J , J' , or S_j are the outermost operation in

the decomposition of the polynomials associated to the Maltsev chain, and that the elements the polynomials are being applied to lie in $e_j(\overline{m}, B)$ for various $j \in \{0, 1, 2\}$ and $\overline{m} \in B^2 \cup B$. Since $u \in e_j(\overline{m}, B)$ and $v \leq u$ implies $v \in e_j(\overline{m}, B)$, our hypothetical Maltsev chain looks something like the diagram below (see Lemma 4.8).



Thus the task of analyzing arbitrary Maltsev chains can be reduced to analyzing two decreasing Maltsev chains of the above form.

Lemmas 4.5 and 4.6 impose a bound on the complexity of the polynomials involved in a typical Maltsev chain (of the form above), and Lemmas 4.11, 4.12, and 4.13 will show that Maltsev chains with associated polynomials of the above form can be reduced to one of 7 different types, each of which is length at most 3 (this is Lemma 4.14). Putting all of this together proves Case 1 above, and is more carefully described in Theorem 4.15.

Cases 2 and 3 are similar to Case 1, but much easier. We first show that if Case 1 does not hold, then all decreasing Maltsev chains must be of length at most 1. That is, if $(r, s) \in \text{Cg}^{\mathbb{B}}(c, d)$ and $s \leq r$ then there is a polynomial $f(x)$ such that $\{f(c), f(d)\} = \{r, s\}$ (this is Lemma 4.16). Next, we show that under some conditions on (c, d) , every polynomial that doesn't collapse $\text{Cg}^{\mathbb{B}}(c, d)$ has a specific nice form (see Lemma 4.18). The way in which (c, d) is produced from (a, b) in Cases 2 and 3 is to show that there is a polynomial that will move $(a(l), b(l))$ into the monolith of each \mathbb{C}_l but not collapse $\text{Cg}^{\mathbb{B}}(a, b)$ (see Lemmas 4.19 and 4.21). Letting (c, d) be the image of (a, b) under this polynomial, we show that (c, d) satisfies the hypotheses of Lemma 4.18 (i.e. all the involved polynomials have a nice form), and then use that $\kappa(\mathcal{V}(\mathbb{A}'(\mathcal{T}))) < \omega$ to bound the complexity of such polynomials. Theorems 4.20 and 4.22 use this reasoning to address Cases 2 and 3.

The last remaining case (Case 4) is short, and is handled in the main theorem itself (Theorem 4.24). The argument there shows that depending on the isomorphism type of \mathbb{C}_k , one of the previous results addresses the case.

We begin the proof for Case 1 with a slightly specialized version of a theorem from Baker and Wang [2].

Lemma 4.1. *Let \mathcal{V} be a locally finite variety and let*

$$P(\overline{c}) = \{p_j(\overline{c}, x_1, x_2, x_3) \mid 1 \leq j \leq K\}$$

be terms in \mathcal{V} with (a fixed number of) constant symbols \overline{c} . Suppose that $J(\overline{c})$ is the set consisting of the Jónsson identities for the polynomials $P(\overline{c})$ in the variables x_1, x_2, x_3 . Then the class

$$\mathcal{M} = \text{Mod}_{\mathcal{V}}(\exists \overline{c} J(\overline{c})) = \{\mathbb{B} \in \mathcal{V} \mid \mathbb{B} \models \exists \overline{c} J(\overline{c})\}$$

has definable principal subcongruences if $\kappa(\mathcal{V}) = N < \omega$.

Proof. The notable modification of the proof given in [2] is at (4.1) below.

Let $\mathbb{B} \in \mathcal{M}$, let $a, b \in \mathbb{B}$ be distinct, and fix $\bar{c} \in B^n$ witnessing $\mathbb{B} \models J(\bar{c})$. Let $\mathbb{B} \leq \prod_{l \in L} \mathbb{C}_l$ be a subdirect representation of \mathbb{B} by subdirectly irreducible algebras. Since $\kappa(\mathcal{V}) < \omega$, each \mathbb{C}_l is finite and there are only finitely many distinct ones. We will construct a finite subalgebra $\mathbb{C} \leq \mathbb{B}$, and then find a pair $(c, d) \in \text{Cg}^{\mathbb{C}}(a, b)$ such that $c \neq d$ and $\text{Cg}^{\mathbb{B}}(c, d)$ is uniformly definable.

Choose $k \in L$ such that $a(k) \neq b(k)$ and $|C_k|$ is maximal with this property. Choose preimage representatives $s_1, \dots, s_M \in B$ of C_k and let

$$(4.1) \quad \mathbb{C} = \langle \{a, b, \bar{c}\} \cup \{s_1, \dots, s_M\} \rangle.$$

Since $\kappa(\mathcal{V}) = N < \omega$ and \mathcal{V} is locally finite, any such \mathbb{C} has size bounded by a number depending only on N and the number of constants \bar{c} . Since \mathbb{C} has bounded size, congruences are defined by a finite number of congruence schemes. By construction, $\pi_k(\mathbb{C}) = \mathbb{C}_k$ and since any subalgebra of \mathbb{B} containing \bar{c} is congruence distributive (any such subalgebra has Jónsson polynomials), \mathbb{C} is congruence distributive.

\mathbb{C}_k is subdirectly irreducible, so $\ker(\pi_k|_{\mathbb{C}})$ is completely meet irreducible in the congruence lattice of \mathbb{C} , $\text{Con}(\mathbb{C})$. Since \mathbb{C} is congruence distributive, $[0, \ker(\pi_k|_{\mathbb{C}})]$ is a prime ideal and therefore the complement is a filter with a least element, call it α , which is join-prime. Therefore α is a principal congruence, say $\alpha = \text{Cg}^{\mathbb{C}}(c, d)$, and α is the least congruence not below $\ker(\pi_k|_{\mathbb{C}})$. Since $\text{Cg}^{\mathbb{C}}(a, b) \not\leq \ker(\pi_k|_{\mathbb{C}})$, by minimality of α we have $\alpha = \text{Cg}^{\mathbb{C}}(c, d) \leq \text{Cg}^{\mathbb{C}}(a, b)$. By the previous paragraph, $|C|$ is bounded by a number depending only on N and the number of constants \bar{c} . It follows that there is a congruence formula determined entirely by this bound that witnesses $(c, d) \in \text{Cg}^{\mathbb{C}}(a, b)$.

Let $l \in L$ and suppose that $c(l) \neq d(l)$. Then $\text{Cg}^{\mathbb{C}}(c, d) \not\leq \ker(\pi_l|_{\mathbb{C}})$ and $a(l) \neq b(l)$. By the minimality of $\alpha = \text{Cg}^{\mathbb{C}}(c, d)$, it must be that $\ker(\pi_l|_{\mathbb{C}}) \leq \ker(\pi_k|_{\mathbb{C}})$. Hence there is a surjective mapping

$$\pi_l(\mathbb{C}) \cong \mathbb{C} / \ker(\pi_l|_{\mathbb{C}}) \twoheadrightarrow \mathbb{C} / \ker(\pi_k|_{\mathbb{C}}) \cong \pi_k(\mathbb{C}) = \mathbb{C}_k.$$

Now, \mathbb{C}_k was chosen to be maximal such that $a(k) \neq b(k)$, so the mapping must also be injective since \mathbb{C}_l is finite. Thus $\pi_l(\mathbb{C}) = \mathbb{C}_k$.

Let $r, s \in \mathbb{B}$ be distinct with $(r, s) \in \text{Cg}^{\mathbb{B}}(c, d)$. We shall construct a finite \mathbb{D} such that $(r, s) \in \text{Cg}^{\mathbb{D}}(c, d)$. Let $\mathbb{D} = \langle C \cup \{r, s\} \rangle$. As with \mathbb{C} , any such \mathbb{D} has size bounded by a number depending only on N and the number of constants \bar{c} , and so congruences in \mathbb{D} are defined by a congruence formula determined entirely by this bound. Since $\bar{c} \in D^n$, we also have that \mathbb{D} is congruence distributive. Let $l \in L$. If $c(l) \neq d(l)$ then by the above paragraph $\pi_l(\mathbb{D}) = \pi_l(\mathbb{C}) = \mathbb{C}_k$, so

$$(r(l), s(l)) \in \text{Cg}^{\pi_l(\mathbb{C})}(c(l), d(l)) = \text{Cg}^{\pi_l(\mathbb{D})}(c(l), d(l)).$$

If $c(l) = d(l)$ then $r(l) = s(l)$, so $(r(l), s(l)) \in \text{Cg}^{\pi_l(\mathbb{D})}(c(l), d(l)) = \mathbf{0}_{\pi_l(\mathbb{D})}$. In either case, $(r(l), s(l)) \in \text{Cg}^{\pi_l(\mathbb{D})}(c(l), d(l))$ for all $l \in L$. To complete the proof we need only prove the following claim.

Claim: Let \mathbb{D} be finite and congruence distributive and let $\mathbb{D} \leq \prod_{i \in I} \mathbb{C}_i$. Then $(r, s) \in \text{Cg}^{\mathbb{D}}(c, d)$ if and only if $(r(i), s(i)) \in \text{Cg}^{\pi_i(\mathbb{D})}(c(i), d(i))$ for all $i \in I$.

Proof of claim: One direction is clear, since the i -th projection map is a homomorphism. For the other direction, we have

$$(r, s) \in \text{Cg}^{\mathbb{D}}(c, d) \vee \ker(\pi_i) \quad \text{for each } i.$$

Since \mathbb{D} is finite, I is finite as well. Therefore by the congruence distributivity of \mathbb{D} ,

$$(r, s) \in \bigwedge_{i \in I} \left(\text{Cg}^{\mathbb{D}}(c, d) \vee \ker(\pi_i) \right) = \text{Cg}^{\mathbb{D}}(c, d) \vee \bigwedge_{i \in I} \ker(\pi_i) = \text{Cg}^{\mathbb{D}}(c, d),$$

as claimed. \square

Define terms e_0 , e_1 , and e_2 in $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ by

$$(4.2) \quad \begin{aligned} e_0(m, x) &= S_0(m, x, x, x), & e_2(m, n, x) &= S_2(m, n, x, x, x), \\ e_1(m, x) &= S_1(m, x, x, x). \end{aligned}$$

Let $\mathcal{V} = \mathcal{V}(\mathbb{A}'(\mathcal{T}))$ and define subclasses of \mathcal{V} ,

$$\mathcal{M}_i = \text{Mod}_{\mathcal{V}}(\exists \overline{m} \, e_i(\overline{m}, x) \approx x) \quad \text{for} \quad i \in \{0, 1, 2\}.$$

We will make use of the fact that if \mathbb{C} is subdirectly irreducible, then either $\mathbb{C} \models \exists \overline{n}[e_i(\overline{n}, x) \approx x]$ for some $i \in \{0, 1, 2\}$ or $\mathbb{C} \models e_i(\overline{y}, x) \approx 0$ for all $i \in \{0, 1, 2\}$. In the case where $\mathbb{C} \models \exists \overline{n}[e_i(\overline{n}, x) \approx x]$, McKenzie [6] proves that since \mathbb{C} is subdirectly irreducible it is necessarily small.

Lemma 4.2. *If \mathcal{T} halts then each \mathcal{M}_i has definable principal subcongruences.*

Proof. Let $i \in \{0, 1, 2\}$. We will show that \mathcal{M}_i satisfies the hypotheses of Lemma 4.1 and thus has definable principal subcongruences. Let $\mathbb{B} \in \mathcal{M}_i$. Choose $\overline{m} \in B^2 \cup B$ witnessing $\mathbb{B} \models e_i(\overline{m}, x) \approx x$. Now, $\mathbb{B} \models e_i(\overline{m}, x) \approx x$ if and only if $\mathbb{B} \models S_i(\overline{m}, x, y, z) \approx (x \wedge y) \vee (x \wedge z)$. Therefore there exists $\overline{m} \in B^2 \cup B$ such that the following

$$(4.3) \quad \begin{aligned} p_0(x, y, z) &= x, & p_1(x, y, z) &= S_i(\overline{m}, x, y, z) = (x \wedge y) \vee (x \wedge z), \\ p_2(x, y, z) &= x \wedge z, & p_3(x, y, z) &= S_i(\overline{m}, z, y, x) = (y \wedge z) \vee (x \wedge z), \\ p_4(x, y, z) &= z, \end{aligned}$$

are polynomials of \mathbb{B} and satisfy the Jónsson identities. If $J_i(\overline{m})$ is the set of Jónsson identities for these polynomials, then $\mathcal{M}_i \subseteq \text{Mod}_{\mathcal{V}}(\exists \overline{m} \, J_i(\overline{m}))$. Since \mathcal{T} halts, $\kappa(\mathcal{V}(\mathcal{M}_i)) \leq \kappa(\mathbb{A}'(\mathcal{T})) < \omega$. By Lemma 4.1, it follows that \mathcal{M}_i has definable principal subcongruences. \square

Let $\Gamma_0^i(w, x, y, z)$ and $\psi_0^i(w, x, y, z)$ be the congruence formulas witnessing definable principal subcongruences for \mathcal{M}_i . Define

$$(4.4) \quad \psi_0(w, x, y, z) = \bigvee_{i=0}^2 \psi_0^i(w, x, y, z) \quad \text{and} \quad \Gamma_0(w, x, y, z) = \bigvee_{i=0}^2 \Gamma_0^i(w, x, y, z).$$

Since Γ_0^i and ψ_0^i are congruence formulas, so are Γ_0 and ψ_0 . Let $\Pi_{\psi_0}(x, y)$ be the formula expressing that the pair (x, y) generates a congruence that is defined by $\psi_0(-, -, x, y)$ in $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ (i.e. the formula asserting that $\psi_0(-, -, x, y)$ is an equivalence relation, is invariant under fundamental translations, and that $\psi_0(x, y, x, y)$ holds). Since each \mathcal{M}_i has definable principal subcongruences and Γ_0 and ψ_0 are the disjunctions of the formulas witnessing DPSC, Γ_0 and ψ_0 witness definable principal subcongruences for the class $\mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3$. In symbols,

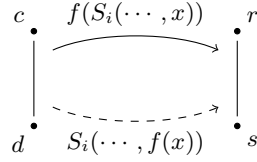
$$\mathcal{M}_i \cup \mathcal{M}_2 \cup \mathcal{M}_3 \models \forall a, b [a \neq b \rightarrow \exists c, d [c \neq d \wedge \Gamma_0(c, d, a, b) \wedge \Pi_{\psi_0}(c, d)]] .$$

The next 5 lemmas provide the groundwork for analyzing the polynomials that make up a hypothetical Maltsev chain. Specifically, they describe the extent to which the non-0-absorbing operations commute with the other operations.

Lemma 4.3. *Let $f(x)$ be a 0-absorbing polynomial and $g(x) = f(S_j(\bar{n}, p, q, x))$ for some $j \in \{0, 1, 2\}$ and some $\bar{n} \in B^2 \cup B$ and $p, q \in B$. If $c, d \in B$ are such that $d \leq c$, then the polynomial*

$$g'(x) = S_j(\bar{n}, g(c), g(d), f(x))$$

satisfies $g'(c) = g(c)$ and $g'(d) = g(d)$.



Proof. Let $\mathbb{B} \leq \prod_{l \in L} \mathbb{C}_l$ be a subdirect representation of \mathbb{B} by subdirectly irreducible algebras and define

$$I = \{l \in L \mid \pi_l(S_j(\bar{n}, p, q, B)) \neq \{0\}\} \quad \text{and} \quad J = L \setminus I.$$

Write a typical $y \in B$ as $y = (y_I, y_J)$, where $y_I = \pi_I(y)$ and $y_J = \pi_J(y)$. Therefore $S_j(\bar{n}, y, y, y) = e_j(\bar{n}, y) = (y_I, 0)$, and so

$$\begin{aligned} g(x) &= f(S_j(\bar{n}, p, q, x)) = f\left(\begin{pmatrix} (p_I \wedge q_I) \vee (p_I \wedge x_{1I}) \\ 0_J \end{pmatrix}\right) \\ &= \left(\begin{pmatrix} f((p_I \wedge q_I) \vee (p_I \wedge x_{1I})) \\ f(0_J) \end{pmatrix}\right) = \left(\begin{pmatrix} f((p_I \wedge q_I) \vee (p_I \wedge x_{1I})) \\ 0_J \end{pmatrix}\right) \in e_j(B) \end{aligned}$$

Thus it suffices to verify the claim on $\pi_I(B)$. For ease of writing, let $r = g(c)$ and $s = g(d)$. Since the operations of $\mathbb{A}'(\mathcal{T})$ are all monotonic, $s \leq r$, and since each \mathbb{C}_l is flat either $s(l) = r(l)$ or $s(l) = 0$. We will examine $g'(x)$ componentwise. Suppose first that $r(l) = s(l)$. Then

$$\begin{aligned} r(l) &= r(l) \wedge s(l) = (r(l) \wedge s(l)) \vee (r(l) \wedge f(x)(l)) \\ &= S_j(\bar{n}, r, s, f(x))(l) = g'(x)(l) \end{aligned}$$

(i.e. if $r(l) = s(l)$ then $g'(\bar{x})(l)$ is constant). Therefore $g'(c)(l) = g'(d)(l) = r(l) = s(l)$. Suppose now that $r(l) \neq s(l)$. Since $s \leq r$, it must be that $s(l) = 0$ and $r(l) \neq 0$. From the definition of $g(x)$ and S_i , it follows that $p(l) \neq q(l)$, $p(l) = c(l)$, and $p(l) \neq d(l)$. Thus $d(l) = 0$. This implies that $S_j(\bar{n}, p, q, c)(l) = p(l) = c(l)$ and $S_j(\bar{n}, p, q, d)(l) = 0 = d(l)$. Therefore

$$\begin{aligned} r(l) &= g(c)(l) = f(S_j(\bar{n}, p, q, c))(l) = f(c)(l) = 0 \vee f(c)(l) \\ &= (r(l) \wedge s(l)) \vee (r(l) \wedge f(c)(l)) = S_j(\bar{n}, r, s, f(c))(l) = g'(c)(l), \text{ and} \\ s(l) &= g(d)(l) = f(S_j(\bar{n}, p, q, d))(l) = f(d)(l) = 0 \vee f(d)(l) \\ &= (r(l) \wedge s(l)) \vee (r(l) \wedge f(d)(l)) = S_j(\bar{n}, r, s, f(c))(l) = g'(d)(l). \end{aligned}$$

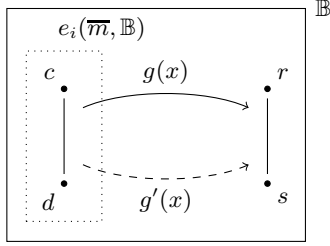
Thus the conclusion of the lemma holds. \square

Lemma 4.4. *Let $c, d \in e_i(\bar{m}, B)$ for some $i \in \{0, 1, 2\}$ and $\bar{m} \in B^2 \cup B$. If $g(x)$ is a polynomial generated by fundamental translations such that $g(d) < g(c)$, then*

there is a polynomial $g'(x)$ with $g'(c) = g(c)$, $g'(d) = g(d)$ and such that one of the following holds: for some constants $p, q \in B$,

- (1) $g'(x) = S_j(\bar{n}, p, q, h(x))$ for some $j \in \{0, 1, 2\}$ and some $\bar{n} \in B^2 \cup B$,
- (2) $g'(x) = f(J(p, q, h(x)))$,
- (3) $g'(x) = f(J'(p, q, h(x)))$, or
- (4) $g'(x) = f(K(p, q, h(x)))$

for some polynomials $h(x)$ and $f(x)$ generated by fundamental translations and such that $h(x)$ is either 0-absorbing or $\text{Range}(h) \subseteq \text{Range}(S_k)$ for some $k \in \{0, 1, 2\}$.



$$\begin{aligned}
 g'(x) &= S_j(\cdots, h(x)), & \text{or} \\
 &= f(J(\cdots, h(x))), & \text{or} \\
 &= f(J'(\cdots, h(x))), & \text{or} \\
 &= f(K(\cdots, h(x)))
 \end{aligned}$$

Proof. Since $c, d \in e_i(\bar{m}, B)$, from Lemma 4.3, if $g(x)$ is a polynomial such that $g(c) \neq g(d)$, then either $g(c), g(d) \in e_i(\bar{m}, B)$ or $g(x)$ is not 0-absorbing. The only fundamental translations which are not 0-absorbing are the S_j in the last 2 variables, and J , J' , and K in the last variable. Since $S_j(\bar{n}, q_1, q_2, x) = S_j(\bar{n}, q_1, x, q_2)$, we will ignore the translations $S_j(\bar{n}, q_1, x, q_2)$.

If $g(c), g(d) \in e_i(\bar{m}, B)$, then the polynomial

$$g'(x) = S_i(\bar{m}, g(c), g(d), e_i(\bar{m}, g(x)))$$

satisfies the conclusion of the lemma. In the case where $g(x)$ is not 0-absorbing, then since $g(x)$ is generated by fundamental translations, it must be that $g(x)$ decomposes as $g(x) = f(F(h(x)))$, where $h(x)$ is a 0-absorbing polynomial generated by fundamental translations and

$$\begin{aligned}
 F(x) \in \{ & S_0(n_1, q_1, q_2, x), S_1(n_1, q_1, q_2, x), S_2(n_0, n_1, q_1, q_2, x), \\
 & J(q_1, q_2, x), J'(q_1, q_2, x), K(q_1, q_2, x) \}
 \end{aligned}$$

(i.e. $F(x)$ is the first non-0-absorbing fundamental translation in the decomposition of $g(x)$). If $F(x)$ is a translation of one of the operations J , J' , or K , then the conclusion clearly holds. If $F(x)$ is a translation of one of the S_j , we will have to proceed by induction on the complexity of $g(x)$.

Assume that the conclusion of the lemma holds for all polynomials of complexity less than $g(x)$. Therefore the lemma holds for $f(x)$ with $c_0 = F(h(c))$ and $d_0 = F(h(d))$ (note that $c_0, d_0 \in e_j(\bar{n}, B)$ since $F(x) = S_j(\bar{n}, q_1, q_2, x)$), so there are polynomials $f_0(x)$ and $h_0(x)$ generated by fundamental translations and such that $h_0(x)$ is 0-absorbing or $\text{Range}(h_0) \subseteq \text{Range}(S_k)$ for some $k \in \{0, 1, 2\}$ and one of

- $g_0(x) = S_j(\bar{n}, p_0, q_0, h_0(x))$ for some $j \in \{0, 1, 2\}$ and some $\bar{n} \in B^2 \cup B$,
- $g_0(x) = f_0(J(p_0, q_0, h_0(x)))$,
- $g_0(x) = f_0(J'(p_0, q_0, h_0(x)))$, or
- $g_0(x) = f_0(K(p_0, q_0, h_0(x)))$

satisfies $g_0(c_0) = f(c_0)$ and $g_0(d_0) = f(d_0)$. Since $c_0 = F(h(c))$, $d_0 = F(h(d))$, and $g(x) = f(F(h(x)))$, we therefore have that one of the polynomials

- $g'(x) = S_j(\bar{n}, p_0, q_0, h_0(F(h(x))))$ for some $j \in \{0, 1, 2\}$ and some $\bar{n} \in B^2 \cup B$,
- $g'(x) = f_0(J(p_0, q_0, h_0(F(h(x))))))$,
- $g'(x) = f_0(J'(p_0, q_0, h_0(F(h(x))))))$, or
- $g'(x) = f_0(K(p_0, q_0, h_0(F(h(x))))))$

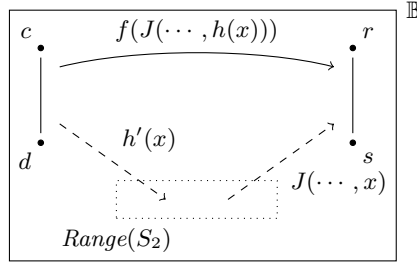
has $g'(c) = g(c)$ and $g'(d) = g(d)$ and $h_0(x)$ is either 0-absorbing or $\text{Range}(h_0) \subseteq \text{Range}(S_k)$ for some $k \in \{0, 1, 2\}$. If $h_0(x)$ is 0-absorbing, then by Lemma 4.3 the conclusion follows since F is a translation of S_j . If $\text{Range}(h_0) \subseteq \text{Range}(S_k)$, then $\text{Range}(h_0 \circ F \circ h) \subseteq \text{Range}(S_k)$ as well, so the conclusion follows in this case as well. \square

The next 3 lemmas quantify the extent to which the non-0-absorbing operations J , J' , and K in the polynomial $g'(x)$ from the conclusion of Lemma 4.4 “commute” with the other fundamental operations.

Lemma 4.5. *Let $h(x)$ and $f(x)$ be polynomials generated by fundamental translations, and $p, q, c, d \in B$ with $d \leq c$. Suppose that $g(x) = f(J(p, q, h(x)))$ with $r = g(c)$, $s = g(d)$, and $r, s \notin \text{Range}(S_i)$ for $i \in \{0, 1, 2\}$. Then there are constants p', q' and a polynomial $h'(x)$ generated by fundamental translations such that the polynomial*

$$g'(x) = J(p', q', h'(x))$$

satisfies $g'(c) = g(c) = r$, $g'(d) = g(d) = s$, and $\text{Range}(h') \subseteq \text{Range}(S_2)$.



Proof. We begin by noting that

$$J(x, y, z) = (x \wedge y) \vee (x \wedge \partial y \wedge z) = (x \wedge y) \vee (x \wedge \partial y \wedge e_2(x, y, z)),$$

from the definition of S_2 (recall $e_2(x, y, z) = S_2(x, y, z, z, z)$) and J . Thus, it will be sufficient to prove that the polynomial $g'(x)$ in the statement of the lemma satisfies $g'(c) = g(c)$ and $g'(d) = g(d)$ without any restrictions on the range of $h'(x)$.

We will prove the claim when $f(x)$ is a fundamental translation. Using induction on the complexity of $g(x)$, the conclusion of the lemma will follow.

Composing the polynomial $J(p, q, h(x))$ with translations of operations from $\{(\cdot), I, T, U_i^1, U_i^2\} \cup \mathcal{L} \cup \mathcal{R}$ produce either constant polynomials or the composition is commutative (i.e. $f(J(p, q, h(x))) = J(f(p), f(q), fh(x))$). Thus the claim holds for these operations.

Case \wedge : We have that $u \wedge J(p, q, h(x)) = J(p, q, h(x)) \wedge u = J(p \wedge u, q, h(x))$.

Case J : The first translation is easy since $J(x, y, z) \wedge w = J(x \wedge w, y, z)$ and $J(x, y, z) \leq J(x, y, x)$. We have

$$\begin{aligned} J(J(p, q, h(x)), u, v) &= J(p, q, h(x)) \wedge J(J(p, q, p), u, v) \\ &= J(p \wedge J(J(p, q, p), u, v), q, h(x)). \end{aligned}$$

For $g(x) = J(u, J(p, q, h(x)), v)$, let

$$g'(x) = J(r, K(r, p, q), S_2(r, K(r, p, q), r, s, g(x))),$$

where $g(d) = s \leq r = g(c)$. Let $\mathbb{B} \leq \prod_{l \in L} \mathbb{C}_l$ be a subdirect representation of \mathbb{B} by subdirectly irreducible algebras. We will show that $g'(c) = r$ and $g'(d) = s$ componentwise. We have that

$$\begin{aligned} g(x) &= (u \wedge p \wedge q) \vee (u \wedge p \wedge \partial q \wedge h(x)) \\ &\quad \vee (u \wedge \partial p \wedge \partial q \wedge v) \vee (u \wedge \partial p \wedge \partial \partial q \wedge \partial h(x)). \end{aligned}$$

The argument at this point breaks down into many cases, depending whether $r(l)$ is equal to $p(l)$, $q(l)$, $\partial p(l)$, or $\partial q(l)$ (if $r(l) \neq 0$, then by the flatness of \mathbb{C}_l it must take on one of these values). The easiest way to keep track of everything is with a table. Since $r(l) = 0$ implies $g'(x)(l) = 0$ and $s(l) = 0$, we will assume that $r(l) \neq 0$. For ease of reading, in the table below we will omit the coordinate when giving values of functions (i.e. “ (l) ” will be omitted from $r(l)$). Additionally, those coordinates which permit $r(l) \neq s(l)$ have been indicated.

r	$K(r, p, q)$	$S_2(r, K(r, p, q), r, s, g(x))$	$g'(x)$	$r \neq s$
$p = q$	$p = r$	0	$r \wedge r$	N
$p = \partial q$	$q = \partial r$	$(r \wedge s) \vee (r \wedge g(x))$	$s \vee (r \wedge g(x))$	Y
$\partial p = \partial q$	$p = \partial r$	$(r \wedge s) \vee (r \wedge g(x))$	$s \vee (r \wedge g(x))$	N
$\partial p = \partial \partial q$	$q = \partial r$	$(r \wedge s) \vee (r \wedge g(x))$	$s \vee (r \wedge g(x))$	Y

Since $r = g(c)$ and $s = g(d)$, we see that $g'(c)(l) = r(l)$ and $g'(d)(l) = s(l)$, except for possibly when $r(l) = p(l) = q(l)$. In this case, however, from the description of $g(x)$ above we see that $g(x)(l)$ is constant, so it must be that

$$r(l) = g(c)(l) = g(d)(l) = s(l).$$

Therefore $g'(c) = r$ and $g'(d) = s$, as claimed.

In the case where $g(x) = J(u, v, J(p, q, h(x)))$, let

$$g'(x) = J(u, v, S_2(u, v, r, s, g(x))).$$

The remarks at the start of the proof show that $g'(c) = g(c) = r$ and $g'(d) = g(d) = s$.

Case J' : The first translation is similar to the case for \wedge , and the second translation is an argument similar to the one requiring the table above. We have

$$\begin{aligned} J'(J(p, q, h(x)), u, v) &= J(p, q, h(x)) \wedge J'(J(p, q, p), u, v) \\ &= J(J'(J(p, q, p), u, v), q, h(x)), \text{ and} \end{aligned}$$

$$J'(u, J(p, q, h(x)), v) = J(J'(u, J(p, q, p), v), J(p, q, h(x)), J'(u, J(p, q, p), v))^\dagger$$

[\dagger : see Case J above]. For $g(x) = J'(u, v, J(p, q, h(x)))$, let

$$g'(x) = J(r, K(r, v, q), S_2(r, K(r, v, q), r, s, g(x))).$$

An argument similar to the one for subcase $J(u, J(p, q, h(x)), v)$ will show that that $g'(c) = g(c) = r$ and $g'(d) = g(d) = s$.

Case S_i : Since $r, s \notin \text{Range}(S_i)$, this case is excluded by hypothesis.

Case K : For $g(x) = K(J(p, q, h(x)), u, v)$, let

$$g'(x) = J(r, K(r, p, q), S_2(r, K(r, p, q), r, s, g(x))).$$

The approach is to take a subdirect representation of \mathbb{B} and prove that $g'(c) = r$ and $g'(d) = s$ componentwise, as in Case J above. Since

$$g(x) = (\partial p \wedge \partial q \wedge u) \vee (\partial p \wedge \partial \partial q \wedge \partial h(x) \wedge u) \vee (\partial p \wedge \partial q \wedge \partial u \wedge v) \\ \vee (\partial p \wedge \partial \partial q \wedge h(x) \wedge \partial u \wedge v) \vee (p \wedge q \wedge u \wedge v) \vee (p \wedge \partial q \wedge h(x) \wedge u \wedge v),$$

the l -th projection of the polynomial $S_2(r, K(r, p, q), r, s, g(x))$ maps $c(l)$ to $r(l)$ and $d(l)$ to $s(l)$ unless $r(l) = (p \wedge q \wedge u \wedge v)(l)$. From the definition of J , it therefore follows that $g'(c) = r$ and $g'(d) = s$.

For the two remaining subcases where either $g(x) = K(u, J(p, q, h(x)), v)$ or $g(x) = K(u, v, J(p, q, h(x)))$, let

$$g'(x) = J(r, K(r, u, q), S_2(r, K(r, u, q), r, s, g(x))).$$

An argument similar to the previous subcase shows that $g'(c) = r$ and $g'(d) = s$.

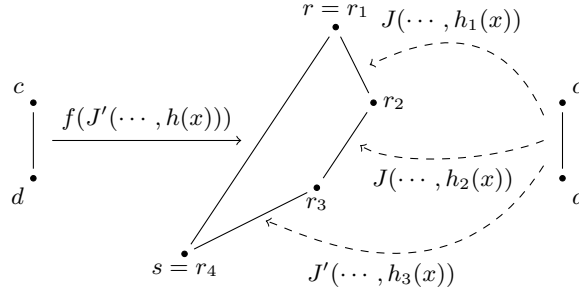
In all cases, the conclusion holds, so by induction on the complexity of $g(x)$, the conclusion holds in general. \square

The above lemma showed for 2 fixed inputs, the J operation can be taken to commute in a very specific way with the other fundamental operations. The situation for J' is much more complicated, requiring a sequence of inputs and a mix of the J and J' operations.

Lemma 4.6. *Let $h(x)$ and $f(x)$ be polynomials generated by fundamental translations with $h(x)$ either 0-absorbing or $\text{Range}(h) \subseteq \text{Range}(S_j)$ for some $j \in \{0, 1, 2\}$, and $p, q, c, d \in B$ with $d \leq c$. Suppose that $g(x) = f(J'(p, q, h(x)))$ with $r = g(c)$, $s = g(d)$, and $r, s \notin \text{Range}(S_i)$ for $i \in \{0, 1, 2\}$. Then there is a decreasing Maltsev chain $r = r_1, r_2, \dots, r_n = s$ connecting r to s with associated polynomials $g_1(x), \dots, g_{n-1}(x)$ of the form*

$$g_k(x) = F_k(p_k, q_k, h_k(x)), \quad F_k \in \{J, J'\}, \quad p_k, q_k \in B,$$

where for each k either $h_k(x)$ is 0-absorbing or $\text{Range}(h_k) \subseteq \text{Range}(S_2)$.



Proof. We will prove the claim when $f(x)$ is a fundamental translation. Using induction on the complexity of $g(x)$ and the result of Lemma 4.5, the conclusion will follow.

Composing the polynomial $J'(p, q, h(x))$ with translations of operations from $\{(\cdot), I, T, U_i^1, U_i^2\} \cup \mathcal{L} \cup \mathcal{R}$ produce either constant polynomials or the composition is commutative (i.e. $f(J'(p, q, h(x))) = J'(f(p), f(q), fh(x))$). Since the operations are 0-absorbing, the claim holds for these operations (either by Lemma 4.3 or since the composition of 0-absorbing polynomials is 0-absorbing).

Case \wedge : We have that $u \wedge J'(p, q, h(x)) = J'(p, q, h(x)) \wedge u = J'(p \wedge u, q, h(x))$.

Case J: The first translation is easy since $J(x, y, z) \wedge w = J(x \wedge w, y, z)$ and $J'(x, y, z) \leq J'(x, y, x)$. We have

$$\begin{aligned} J(J'(p, q, h(x)), u, v) &= J'(p, q, h(x)) \wedge J(J'(p, q, p), u, v) \\ &= J'(p \wedge J(J'(p, q, p), u, v), q, h(x)). \end{aligned}$$

For $g(x) = J(u, J'(p, q, h(x)), v)$ we must introduce a new “link” in our Maltsev chain. Let

$$\begin{aligned} g_1(x) &= J(r, K(r, p, q), S_2(r, K(r, p, q), r, s, g(x))) & \text{and} \\ g_2(x) &= J'(t_1, K(t_1, p, q), h(x)), & \text{where } t_1 = g_1(d) \end{aligned}$$

(recall that $r = g(c)$ and $s = g(d)$). We have that

$$\begin{aligned} g(x) &= (u \wedge p \wedge q \wedge h(x) \wedge v) \vee (u \wedge p \wedge \partial q \wedge v) \\ &\quad \vee (u \wedge \partial p \wedge \partial q \wedge \partial h(x)) \vee (u \wedge \partial p \wedge \wedge q). \end{aligned}$$

The argument at this point breaks down into many cases, depending whether $r(l)$ is equal to $p(l)$, $q(l)$, $\partial p(l)$, or $\partial q(l)$ (if $r(l) \neq 0$, then by the flatness of \mathbb{C}_l it must take on one of these values). The easiest way to keep track of everything is with a table. Since $r(l) = 0$ implies $g'(x)(l) = 0$ and $s(l) = 0$, we will assume that $r(l) \neq 0$. For ease of reading, in the table below we will omit the coordinate when giving values of functions (i.e. “ (l) ” will be omitted from $r(l)$). Additionally, those coordinates which permit $r(l) \neq s(l)$ have been indicated.

r	$K(r, p, q)$	$S_2(r, K(r, p, q), r, s, g(x))$	$g_1(x)$	$r \neq s$
$p = q$	$p = r$	0	$r \wedge r$	Y
$p = \partial q$	$q = \partial r$	$(r \wedge s) \vee (r \wedge g(x))$	$s \vee (r \wedge g(x))$	N
$\partial p = \partial q$	$p = \partial r$	$(r \wedge s) \vee (r \wedge g(x))$	$s \vee (r \wedge g(x))$	Y
$\partial p = \partial \partial q$	$q = \partial r$	$(r \wedge s) \vee (r \wedge g(x))$	$s \vee (r \wedge g(x))$	N

Since $r = g(c)$ and $s = g(d)$, we have that $g_1(c) = r$, and $t_1(l) = g_1(d)(l) = s(l)$ in all cases except for possibly when $r(l) = p(l) = q(l)$. It follows that $s \leq t_1 \leq r$. We will now show (with another similar table) that $g_2(c) = t_1$ and $g_2(d) = s$. The first column of the table below corresponds to the 2nd to last column of the table above evaluated at $x = d$.

t_1	$K(t_1, p, q)$	$g_2(x)$	$t_1 \neq s$
$r = p = q$	$p = t_1$	$t_1 \wedge h(x)$	Y
$r = s = p = \partial q$	$q = \partial t_1$	t_1	N
s	$(p \wedge \partial s) = \partial t_1$	t_1	N
$r = s = \partial p = \partial \partial q$	$q = \partial t_1$	t_1	N

From the table we can see that $g_2(c)(l) = t_1(l)$ in all cases except for possibly when $r(l) = p(l) = q(l)$. In this case, from the definition of $g(x)$ we have that $r(l) = h(c)(l)$, so $g_2(c)(l) = t_1(l)$ (the previous table indicates that $t_1(l) = r(l)$ when $r(l) = p(l) = q(l)$). Therefore $g_2(c) = t_1$. Since $t_1(l)$ differs from $s(l)$ only when $r(l) = p(l) = q(l)$, and since in this case $h(d)(l) = s(l)$ (from the definition of $g(x)$ at the start of the case), it follows that $g_2(d) = s$.

In the case where $g(x) = J(u, v, J'(p, q, h(x)))$, let

$$g_1(x) = J(u, v, S_2(u, v, r, s, h(x))).$$

An argument similar to many in Lemma 4.5 will show that $g_1(c) = g(c) = r$ and $g_1(d) = g(d) = s$.

Case J' : We have

$$\begin{aligned} J'(J'(p, q, h(x)), u, v) &= J'(p, q, h(x)) \wedge J'(J(p, q, p), u, v) \\ &= J'(J'(J(p, q, p), u, v), q, h(x)) \end{aligned}$$

For $g(x) = J'(u, J'(p, q, h(x)), v)$ we must again introduce a new “link” in our Maltsev chain. Let

$$\begin{aligned} g_1(x) &= J(r, K(r, p, q), S_2(r, K(r, p, q), r, s, g(x))) & \text{and} \\ g_2(x) &= J'(t_1, K(t_1, p, q), h(x)), & \text{where } t_1 = g_1(d). \end{aligned}$$

An argument similar to the corresponding subcase of Case J will show that $g_1(c) = r$, $g_1(d) = g_2(c) = t_1$, and $g_2(d) = s$. For $g(x) = J'(u, v, J'(p, q, h(x)))$, let

$$g_1(x) = J'(r, K(r, v, q), h(x)).$$

An argument similar to the one in Case J above will show that $g_1(c) = r$ and $g_2(d) = s$.

Case S_i : Since $r, s \notin \text{Range}(S_i)$, this case is excluded by hypothesis.

Case K : For $g(x) = K(J'(p, q, h(x)), u, v)$, let

$$\begin{aligned} g_1(x) &= J(r, K(r, p, q), S_2(r, K(r, p, q), r, s, g(x))) & \text{and} \\ g_2(x) &= J'(t_1, K(t_1, p, q), h(x)), & \text{where } t_1 = g_1(d). \end{aligned}$$

An argument similar to the one in Case J shows that $g_1(c) = r$, $g_1(d) = g_2(c) = t_1$, and $g_2(d) = s$. For the two remaining subcases where we have either $g(x) = K(u, J'(p, q, h(x)), v)$ or $g(x) = K(u, v, J'(p, q, h(x)))$, let

$$\begin{aligned} g_1(x) &= J(r, K(r, u, p), S_2(r, K(r, u, q), r, s, g(x))) & \text{and} \\ g_2(x) &= J'(t_1, K(t_1, u, q), h(x)), & \text{where } t_1 = g_1(d). \end{aligned}$$

The usual argument will show that $g_1(c) = r$, $g_1(d) = g_2(c) = t_1$, and $g_2(d) = s$.

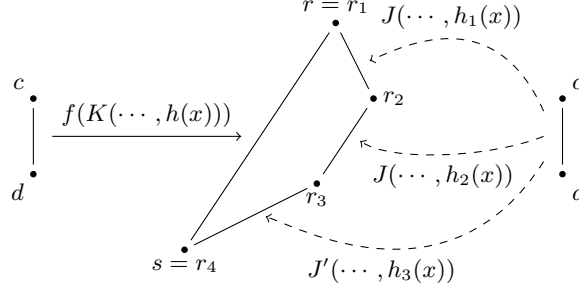
In all cases, the conclusion holds, so by Lemma 4.5 and induction on the complexity of $g(x)$, the conclusion holds in general. \square

In the next lemma, we see that the K operation behaves essentially the same as the J' operation.

Lemma 4.7. *Let $h(x)$ and $f(x)$ be polynomials generated by fundamental translations with $h(x)$ either 0-absorbing or $\text{Range}(h) \subseteq \text{Range}(S_j)$ for some $j \in \{0, 1, 2\}$, and $p, q, c, d \in B$ with $d \leq c$. Suppose that $g(x) = f(K(p, q, h(x)))$ with $r = g(c)$, $s = g(d)$, and $r, s \notin \text{Range}(S_i)$ for $i \in \{0, 1, 2\}$. Then there is a decreasing Maltsev chain $r = r_1, r_2, \dots, r_n = s$ connecting r to s with associated polynomials $g_1(x), \dots, g_{n-1}(x)$ of the form*

$$g_k(x) = F_k(p_k, q_k, h_k(x)), \quad F_k \in \{J, J'\}, \quad p_k, q_k \in B,$$

where for each k either $h_k(x)$ is 0-absorbing or $\text{Range}(h_k) \subseteq \text{Range}(S_2)$.



Proof. Let $g(x) = K(p, q, h(x))$ (i.e. take $f(x) = x$ in the statement) and

$$\begin{aligned} g_1(x) &= J(r, K(r, p, q), S_2(r, K(r, p, q), r, s, h(x))) & \text{and} \\ g_2(x) &= J'(t_1, K(t_1, p, q), h(x)), & \text{where } t_1 = g_1(d). \end{aligned}$$

We will show that $r = g_1(c)$, $t_1 = g_2(c)$, and $s = g_2(d)$. Using Lemmas 4.5 and 4.6, the conclusion will follow.

We have

$$g(x) = K(p, q, h(x)) = (\partial p \wedge q) \vee (\partial p \wedge q \wedge h(x)) \vee (p \wedge q \wedge h(x)).$$

Let $\mathbb{B} \leq \prod_{l \in L} \mathbb{C}_l$ be a subdirect representation of \mathbb{B} by subdirect irreducibles. As usual, we will analyze $g_1(x)$ componentwise, and it will be easiest to make a table. Since $r(l) = 0$ implies $g_1(x)(l) = 0$ and $s(l) = 0$, we will assume that $r(l) \neq 0$. For ease of reading, in the table below we will omit the coordinate when giving values of functions (i.e. “ (l) ” will be omitted from $r(l)$). Additionally, those coordinates which permit $r(l) \neq s(l)$ have been indicated.

r	$K(r, p, q)$	$S_2(r, K(r, p, q), r, s, h(x))$	$g_1(x)$	$r \neq s$
$\partial p = q$	$p = \partial r$	$(r \wedge s) \vee (r \wedge h(x))$	$s \wedge (r \wedge h(x))$	N
$\partial p = \partial q$	$p = \partial r$	$(r \wedge s) \vee (r \wedge h(x))$	$s \wedge (r \wedge h(x))$	Y
$p = q$	$p = r$	0	r	Y

Since $r = g(c)$ and $s = g(d)$, we have that $g_1(c) = r$ and $t_1(l) = g_1(d)(l) = s(l)$ in all cases except for possibly when $r(l) = p(l) = q(l)$. It follows that $s \leq t_1 \leq r$. We will now show (with another similar table) that $g_2(c) = t_1$ and $g_2(d) = s$.

t_1	$K(t_1, p, q)$	$g_2(x)$	$t_1 \neq s$
$r = s = \partial p = q$	$p = \partial t_1$	t_1	N
s	$(p \wedge \partial s) = \partial t_1$	t_1	N
$r = p = q$	$p = t_1$	$t_1 \wedge h(x)$	Y

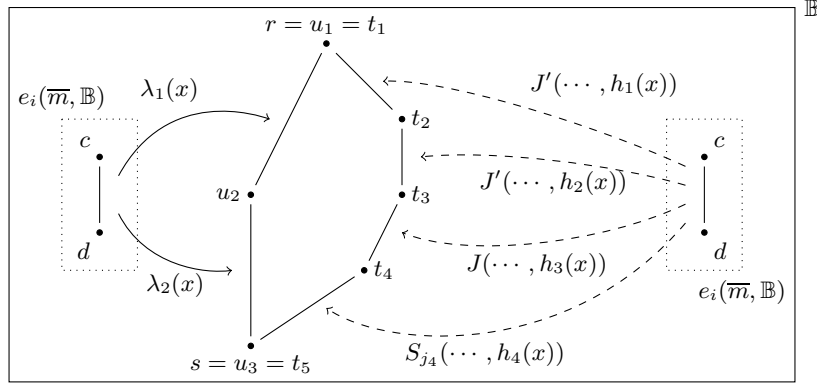
From the table we see that $g_2(c)(l) = t_1(l)$ in all cases except for possibly when $r(l) = p(l) = q(l)$. In this case, from the definition of $g(x)$ we have that $r(l) = h(c)(l)$, so $g_2(c)(l) = t_1(l)$ in this case as well (the previous table indicates that $t_1(l) = r(l)$ when $r(l) = p(l) = q(l)$). Therefore $g_2(c) = t_1$. Since $t_1(l)$ differs from $s(l)$ only when $r(l) = p(l) = q(l)$, and since in this case $h(d)(l) = s(l)$, we have that $g_2(d) = s$. \square

Lemma 4.8. *Let $c, d \in e_i(\overline{m}, B)$ for some $i \in \{0, 1, 2\}$ and $\overline{m} \in B^2 \cup B$ with $d \leq c$. Suppose that $(r, s) \in Cg^{\mathbb{B}}(c, d)$ with $s \leq r$ is witnessed by a decreasing Maltsev sequence, say $r = u_1, \dots, u_n = s$, with associated polynomials generated from fundamental translations $\lambda_1(x), \dots, \lambda_{n-1}(x)$. Then there is another decreasing*

Maltsev sequence, $r = t_1, \dots, t_m = s$, with associated polynomials generated from fundamental translations $g_1(x), \dots, g_{m-1}(x)$ such that for each $k \in \{1, \dots, m-1\}$, one of the following holds:

- (1) $g_k(x) = S_{j_k}(\overline{m}_k, t_k, t_{k+1}, h_k(x))$ for some $j_k \in \{0, 1, 2\}$ and $\overline{m}_k \in B^2 \cup B$,
- (2) $g_k(x) = J(t_k, q_k, h_k(x))$, or
- (3) $g_k(x) = J'(t_k, q_k, h_k(x))$,

for some constants $q_k \in B$ and polynomials $h_k(x)$ such that either $\text{Range}(h_k) \subseteq \text{Range}(S_{l_k})$ for some $l_k \in \{0, 1, 2\}$ or $h_k(x)$ is 0-absorbing.



Proof. Select a consecutive pair, u_k and u_{k+1} from the Maltsev sequence. We will show that the claim holds for the pair, and therefore it must hold for the entire sequence. By Lemma 4.4, we can assume that one of the following holds:

- (1) $\lambda_k(x) = S_j(\overline{n}, p, q, h(x))$ for some $j \in \{0, 1, 2\}$ and some $\overline{n} \in B^2 \cup B$,
- (2) $\lambda_k(x) = f(J(p, q, h(x)))$,
- (3) $\lambda_k(x) = f(J'(p, q, h(x)))$, or
- (4) $\lambda_k(x) = f(K(p, q, h(x)))$,

for constants $p, q \in B$ and polynomials generated by fundamental translations $f(x)$ and $h(x)$ such that $h(x)$ is 0-absorbing or $\text{Range}(h) \subseteq \text{Range}(S_l)$ for some $l \in \{0, 1, 2\}$. In the first possibility, since $u_{k+1} \leq u_k$, $S_j(\overline{n}, u_k, u_{k+1}, e_j(\overline{n}, h(c))) = \lambda_k(c)$ and $S_j(\overline{n}, u_k, u_{k+1}, e_j(\overline{n}, h(d))) = \lambda_k(d)$. Therefore, in this case take $g_k(x) = S_j(\overline{n}, u_k, u_{k+1}, e_j(\overline{n}, h(x)))$.

In the remaining 3 possibilities, we apply Lemmas 4.5, 4.6, and 4.7 to the pair u_k, u_{k+1} to get a decreasing Maltsev sequence $u_k = t_k, t_{k+1}, \dots, t_{k+m'} = u_{k+1}$ with associated polynomials generated by fundamental translations $g_k(x), \dots, g_{k+m'-1}(x)$ such that for all $l \in \{k, \dots, k+m'-1\}$,

$$g_l(x) = F_l(p_l, q_l, h_l(x)) \quad \text{for} \quad F_l \in \{J, J'\} \text{ and } p_l, q_l \in B$$

and either $\text{Range}(h_l) \subseteq \text{Range}(S_2)$ or $h_l(x)$ is 0-absorbing.

Finally, we observe that if $f(x) = F(p, q, h(x))$ is a polynomial with $F \in \{J, J'\}$ and $f(d) \leq f(c)$, then

$$f(c) = F(f(c), q, h(c)) \quad \text{and} \quad f(d) = F(f(c), q, h(d)).$$

Applying this observation to the $g_l(x)$ and using the fact that $t_k, \dots, t_{k+m'}$ is a decreasing sequence completes the proof. \square

At this point, we have established the tools necessary to transform general decreasing Maltsev chains into longer chains whose associated polynomials are of a

very specific form. Now, we move to on to show that these longer chains can be shortened and come in just 7 types, and that these 7 different types of chains are definable. The following definition simplifies the discussion.

Definition 4.9. Let $r_1, \dots, r_n \in B$ be a sequence of elements. We write

$$r_1 \xrightarrow{F_1} r_2 \xrightarrow{F_3} r_3 \cdots r_{n-1} \xrightarrow{F_{n-1}} r_n$$

for $F_i \in \{J, J', S_0, S_1, S_2\}$ if

- (1) for $F_i \in \{J, J'\}$, there exist constants $p_i, q_i \in B$ and $\bar{n}_i \in B^2 \cup B$ such that

$$r_i = F_i(p_i, q_i, e_{j_i}(\bar{n}_i, r_i)) \quad \text{and} \quad r_{i+1} = F_i(p_i, q_i, e_{j_i}(\bar{n}_i, r_{i+1}))$$

for some $j_i \in \{0, 1, 2\}$.

- (2) for $F_i \in \{S_0, S_1, S_2\}$ there exists $\bar{n}_i \in B^2 \cup B$ such that

$$r_i = F_i(\bar{n}_i, r_i, r_i, r_i) \quad \text{and} \quad r_{i+1} = F_i(\bar{n}_i, r_{i+1}, r_{i+1}, r_{i+1}).$$

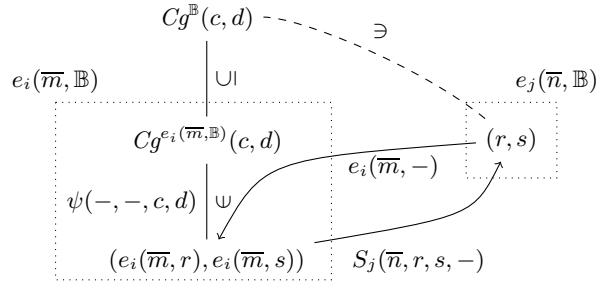
Such a sequence will be referred to as an F_1 - F_2 - \dots - F_{n-1} chain. If it is the case that for all i , $(r_i, r_{i+1}) \in \text{Cg}^{\mathbb{B}}(c, d)$ for some $c, d \in B$, then we will say that $(r_1, r_n) \in \text{Cg}^{\mathbb{B}}(c, d)$ is *witnessed* by an F_1 - \dots - F_{n-1} chain.

Lemma 4.10. Let $c, d \in e_i(\bar{m}, B)$ for some $i \in \{0, 1, 2\}$ and $\bar{m} \in B^2 \cup B$ and assume that the congruence formula $\psi(-, -, c, d)$ defines $\text{Cg}^{e_i(\bar{m}, \mathbb{B})}(c, d)$ in $e_i(\bar{m}, \mathbb{B})$. Suppose that $r, s \in e_j(\bar{n}, B)$ for some $j \in \{0, 1, 2\}$ and $\bar{n} \in B^2 \cup B$ with $s \leq r$. Then $(r, s) \in \text{Cg}^{\mathbb{B}}(c, d)$ if and only if

$$\mathbb{B} \models \psi(e_i(\bar{m}, r), e_i(\bar{m}, s), c, d)$$

and $r = S_j(\bar{n}, r, s, e_i(\bar{m}, r))$ and $s = S_j(\bar{n}, r, s, e_i(\bar{m}, s))$.

In the terminology of Definition 4.9, every $(r, s) \in \text{Cg}^{\mathbb{B}}(c, d)$ with $r, s \in e_i(\bar{n}, B)$ for some $i \in \{0, 1, 2\}$ and $\bar{m} \in B^2 \cup B$ is witnessed by an S_j chain.



Proof. Suppose first that $(r, s) \in \text{Cg}^{\mathbb{B}}(c, d)$ and let $\mathbb{B} \leq \prod_{l \in L} \mathbb{C}_l$ be a subdirect representation of \mathbb{B} by subdirectly irreducible algebras in $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$. Define

$$I = \{l \in L \mid e_i(\bar{m}, B)(l) \neq \{0\}\} \quad \text{and} \quad J = L \setminus I,$$

and write a typical element $x \in B$ as $x = (x_I, x_J)$, where $x_I \in \pi_I(B)$ and $x_J \in \pi_J(B)$. Since $c, d \in e_i(\bar{m}, B)$, we have $c = (c_I, 0_J)$ and $d = (d_I, 0_J)$. Hence, if $(r, s) \in \text{Cg}^{\mathbb{B}}(c, d)$, it must be that $r = (r_I, z_J)$ and $s = (s_I, z_J)$ (i.e. $\pi_J(r) = \pi_J(s)$).

From the definition of S_i , we have that $e_i(\bar{m}, -)$ is a homomorphism from \mathbb{B} to $e_i(\bar{m}, \mathbb{B})$. Therefore $(e_i(\bar{m}, r), e_i(\bar{m}, s)) \in \text{Cg}^{e_i(\bar{m}, \mathbb{B})}(c, d)$, and

$$\mathbb{B} \models \psi(e_i(\bar{m}, r), e_i(\bar{m}, s), c, d),$$

since ψ is existentially quantified (it is a congruence formula) and $e_i(\overline{m}, \mathbb{B}) \leq \mathbb{B}$. Now since $r \in e_j(\overline{n}, B)$, if $t \leq r$ then $t \in e_j(\overline{n}, B)$. Therefore $e_i(\overline{m}, r), e_i(\overline{m}, s) \in e_j(\overline{n}, B)$. Thus

$$\begin{aligned} S_j(\overline{n}, r, s, e_i(\overline{m}, r)) &= S_j\left(\overline{n}, \begin{pmatrix} r_I \\ z_J \end{pmatrix}, \begin{pmatrix} s_I \\ z_J \end{pmatrix}, \begin{pmatrix} r_I \\ 0 \end{pmatrix}\right) \\ &= \left(\begin{pmatrix} r_I \\ z_J \end{pmatrix} \wedge \begin{pmatrix} s_I \\ z_J \end{pmatrix}\right) \vee \left(\begin{pmatrix} r_I \\ z_J \end{pmatrix} \wedge \begin{pmatrix} r_I \\ 0 \end{pmatrix}\right) = \begin{pmatrix} s_I \\ z_J \end{pmatrix} \vee \begin{pmatrix} r_I \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} r_I \\ z_J \end{pmatrix} = r, \text{ and likewise} \\ S_j(\overline{n}, r, s, e_i(\overline{m}, s)) &= S_j\left(\overline{n}, \begin{pmatrix} r_I \\ z_J \end{pmatrix}, \begin{pmatrix} s_I \\ z_J \end{pmatrix}, \begin{pmatrix} s_I \\ 0 \end{pmatrix}\right) \\ &= \left(\begin{pmatrix} r_I \\ z_J \end{pmatrix} \wedge \begin{pmatrix} s_I \\ z_J \end{pmatrix}\right) \vee \left(\begin{pmatrix} r_I \\ z_J \end{pmatrix} \wedge \begin{pmatrix} s_I \\ 0 \end{pmatrix}\right) = \begin{pmatrix} s_I \\ z_J \end{pmatrix} \vee \begin{pmatrix} s_I \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} s_I \\ z_J \end{pmatrix} = s, \end{aligned}$$

completing the forward direction.

Suppose now that $\mathbb{B} \models \psi(e_i(\overline{m}, r), e_i(\overline{m}, s), c, d)$, and $r = S_j(\overline{n}, r, s, e_i(\overline{m}, r))$ and $s = S_j(\overline{n}, r, s, e_i(\overline{m}, s))$. Since ψ is a congruence formula and $e_i(\overline{m}, -)$ is a homomorphism from \mathbb{B} to $e_i(\overline{m}, \mathbb{B})$ and $c, d \in e_i(\overline{m}, B)$, we have

$$e_i(\overline{m}, \mathbb{B}) \models \psi(e_i(\overline{m}, r), e_i(\overline{m}, s), c, d).$$

Thus, $(e_i(\overline{m}, r), e_i(\overline{m}, s)) \in \text{Cg}^{e_i(\overline{m}, \mathbb{B})}(c, d) \subseteq \text{Cg}^{\mathbb{B}}(c, d)$. By hypothesis, we also have that $r = S_j(\overline{n}, r, s, e_i(\overline{m}, r))$ and $s = S_j(\overline{n}, r, s, e_i(\overline{m}, s))$, so it follows that $(r, s) \in \text{Cg}^{\mathbb{B}}(c, d)$. \square

In light of the above lemma, define

$$\begin{aligned} \psi_S(w, x, y, z) &= \bigvee_{i=0}^2 \bigvee_{j=0}^2 [\exists \overline{n} [y = e_i(\overline{n}, y) \wedge z = e_i(\overline{n}, z) \wedge \psi_0(e_i(\overline{n}, w), e_i(\overline{n}, x), y, z)]] \\ (4.5) \quad &\wedge \exists \overline{m} [w = S_j(\overline{m}, w, x, e_j(\overline{m}, w)) \wedge x = S_j(\overline{m}, w, x, e_j(\overline{m}, x))]. \end{aligned}$$

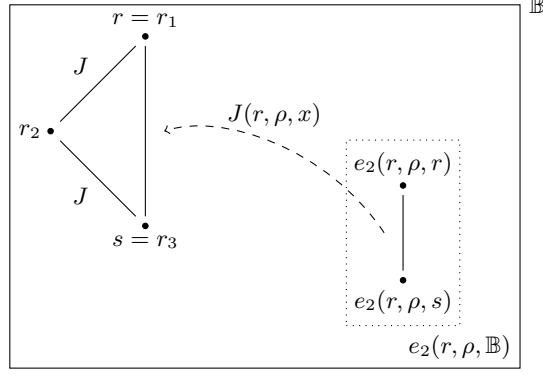
If c, d, r, s satisfy the hypotheses of Lemma 4.10, then $(r, s) \in \text{Cg}^{\mathbb{B}}(c, d)$ if and only if $\mathbb{B} \models \psi_S(r, s, c, d)$.

Lemma 4.11. *Suppose that the decreasing sequence $r = r_1, r_2, \dots, r_n = s \in B$ is such that $(r_i, r_{i+1}) \in \text{Cg}^{\mathbb{B}}(c, d)$ for some $c, d \in B$ and there are constants $p_1, q_1, \dots, p_{n-1}, q_{n-1} \in B$ such that*

$$r_i = J(p_i, q_i, r_i) \quad \text{and} \quad r_{i+1} = J(p_i, q_i, r_{i+1})$$

for $1 \leq i \leq n-1$. Then there exists a constant $\rho \in B$ such that $r = J(r, \rho, r')$ and $s = J(r, \rho, s')$, where $r' = e_2(r, \rho, r)$ and $s' = e_2(r, \rho, s)$.

Since $J(a, b, c) = J(a, b, e_2(a, b, c))$ (from the definition of J), without loss of generality we can take $r_i \in e_2(p_i, q_i, \mathbb{B})$. Therefore, in the terminology of Definition 4.9, for each J - J -...- J chain (of any length), there is a J chain with the same endpoints.



Proof. First, observe that since the chain is decreasing and $r \geq s$, if we replace q_i with $J(q_i, p_i, q_i)$, then we can replace each p_i with r . Thus, we may assume that

$$r_i = J(r, q_i, r_i) \quad \text{and} \quad r_{i+1} = J(r, q_i, r_{i+1}).$$

The proof shall be by induction on n (the length of the chain). If $n = 1$, then

$$r = J(r, q_1, r) \quad \text{and} \quad s = J(r, q_1, s).$$

Therefore

$$r = (r \wedge \partial q_1 \wedge r) \vee (r \wedge q_1) \quad \text{and} \quad s = (r \wedge \partial q_1 \wedge s) \vee (r \wedge q_1).$$

Hence without loss of generality, we can replace the last occurrence of r in $r = J(r, q_1, r)$ with $r' = e_2(r, q_1, r)$, and the last occurrence of s in $s = J(r, q_1, s)$ with $s' = e_2(r, q_1, s)$. After making these replacements, the conclusion of the lemma follows with $\rho = q_1$.

Assume now that the lemma holds for all chains of length less than N , and consider a chain of length N : $r = r_1, \dots, r_N = s$. Applying the inductive hypothesis to the subchain $r = r_1, \dots, r_{N-1}$, there exists $\rho_1 \in \mathbb{B}$ with $r = J(r, \rho_1, r'')$ and $r_{N-1} = J(r, \rho_1, r''_{N-1})$, where $r'' = e_2(r, \rho_1, r)$ and $r''_{N-1} = e_2(r, \rho_1, r_{N-1})$. We therefore have

$$(4.6) \quad \begin{aligned} r &= J(r, \rho_1, r''), & r_{N-1} &= J(r, \rho_1, r''_{N-1}) = J(r, q_{N-1}, r_{N-1}), \\ s &= J(r, q_{N-1}, s). \end{aligned}$$

Let $\rho = K(r, \rho_1, q_{N-1})$, $r' = e_2(r, \rho, r)$, and $s' = e_2(r, \rho, s)$. We will now show that $r = J(r, \rho, r')$ and $s = J(r, \rho, s')$, proving the lemma.

Let $\mathbb{B} = \prod_{l \in L} \mathbb{C}_l$ be a subdirect representation of \mathbb{B} by subdirectly irreducible algebras. We will analyze the polynomial $J(r, \rho, x)$ coordinatewise, and as usual it will be easiest to use a table. Before the table is constructed, however, we will determine which coordinates permit $r(l) \neq s(l)$. Since $r \geq r_{N-1} \geq s$, either $r(l) \neq r_{N-1}(l) = s(l) = 0$, or $r(l) = r_{N-1}(l) \neq s(l) = 0$. The equalities (4.6) give us

$$\begin{aligned} r &= (r \wedge \rho_1) \vee (r \wedge \partial \rho_1 \wedge r''), & r_{N-1} &= (r \wedge \rho_1) \vee (r \wedge \partial \rho_1 \wedge r''_{N-1}), \\ r_{N-1} &= (r \wedge q_{N-1}) \vee (r \wedge \partial q_{N-1} \wedge r_{N-1}), & s &= (r \wedge q_N) \vee (r \wedge \partial q_N \wedge s). \end{aligned}$$

Observe that $r(l) = \partial \rho_1(l)$ implies $r(l) = e_2(r, \partial \rho_1, r)(l) = r''(l)$. Assume first that $r(l) \neq r_{N-1}(l) = s(l) = 0$. In this case, it must be that $r(l) = \partial \rho_1(l)$ and $r(l) = \partial q_{N-1}(l)$. Assume now that $r(l) = r_{N-1}(l) \neq s(l) = 0$. In this case, it must be that $r(l) = r_{N-1}(l) \in \{\rho_1, \partial \rho_1\}$ and $r = \partial q_{N-1}$. We now assemble all of this in

the table below. As usual, since $r(l) = 0$ implies $r_{N-1}(l) = s(l) = 0$, we assume that $r(l) \neq 0$. In particular, this means that $r(l) \in \{\rho_1(l), \partial\rho_1(l)\}$.

r	$\rho = K(r, \rho_1, q_{N-1})$	$J(r, \rho, e_2(r, \rho, x))$	$r \neq s$
$\rho_1 = q_{N-1}$	r	r	N
$\rho_1 = \partial q_{N-1}$	$q_{N-1} = \partial r$	$e_2(r, \partial r, x)$	Y
$\partial\rho_1 = q_{N-1}$	$\rho_1 = \partial r$	$e_2(r, \partial r, x)$	N
$\partial\rho_1 = \partial q_{N-1}$	$\rho_1 = \partial r$	$e_2(r, \partial r, x)$	Y

If $r(l) = \partial\rho(l)$, then $r'(l) = e_2(r, \rho, r)(l) = r(l)$ and $s'(l) = e_2(e, \rho, s)(l) = s(l)$. Therefore the table above show that $J(r, \rho, r') = r$ and $J(r, \rho, s') = s$. \square

In light of the above lemma, define

$$(4.7) \quad \psi_J(w, x, y, z) = \exists b [\psi_S(e_2(w, b, w), e_2(w, b, x), y, z) \\ \wedge w = J(w, b, e_2(w, b, w)) \wedge x = J(w, b, e_2(w, b, x))]$$

(ψ_S was defined in (4.5)). If $c, d \in e_i(\bar{n}, B)$ for some $i \in \{0, 1, 2\}$ and some $\bar{n} \in B^2 \cup B$ and $(r, s) \in \text{Cg}^{\mathbb{B}}(c, d)$ is witnessed by a chain satisfying the hypotheses of Lemma 4.11, then $(r, s) \in \text{Cg}^{\mathbb{B}}(c, d)$ if and only if $\mathbb{B} \models \psi_J(r, s, c, d)$.

Lemma 4.12. *Suppose that the decreasing sequence $r = r_1, r_2, \dots, r_n = s \in B$ is such that $(r_i, r_{i+1}) \in \text{Cg}^{\mathbb{B}}(c, d)$ for some $c, d \in B$ and there are constants $p_1, q_1, \dots, p_{n-1}, q_{n-1} \in B$ such that*

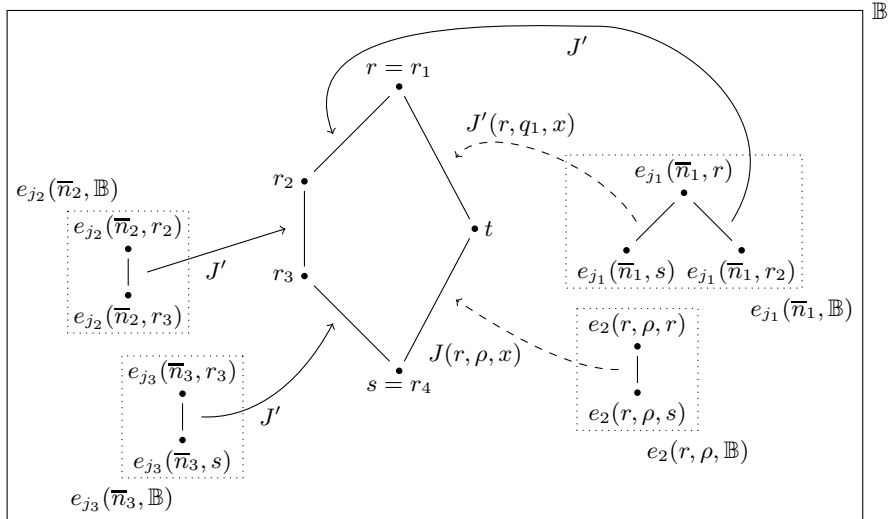
$$r_i = J'(p_i, q_i, e_{j_i}(\bar{n}_i, r_i)), \quad r_{i+1} = J'(p_i, q_i, e_{j_i}(\bar{n}_i, r_{i+1}))$$

for some $j_i \in \{0, 1, 2\}$ and $\bar{n}_i \in B^2 \cup B$. Then there exist constants, $\rho, t \in B$ such that

$$r = J'(r, q_1, e_{j_1}(\bar{n}_1, r)), \quad t = J'(r, q_1, e_{j_1}(\bar{n}_1, s)) = J(t, \rho, r'), \\ s = J(t, \rho, s'),$$

where $r' = e_2(r, \rho, r)$, and $s' = e_2(r, \rho, s)$.

In the terminology of Definition 4.9, for every $J' \dots J'$ chain (of arbitrary length) there is a J' - J chain with the same endpoints.



Proof. The proof shall be by induction on n (the length of the sequence). If $n = 1$, the lemma is trivially true. Assume now that the lemma holds for all sequences of length less than N , and consider a sequence of length N : $r = r_1, \dots, r_N = s$. Apply the inductive hypothesis to the subsequence $r_2, \dots, r_N = s$ to get

$$\begin{aligned} r_2 &= J'(r_2, q_2, e_{j_2}(\bar{n}_2, r_2)), & t_1 &= J'(r_2, q_2, e_{j_2}(\bar{n}_2, s)) = J(t_1, \rho_1, r'_2) \\ s &= J(t_1, \rho_1, s'), & \text{where } r'_2 &= e_2(r_2, \rho, r_2) \text{ and } s' = e_2(r_2, \rho, s) \end{aligned}$$

for some constants $\rho_1, t_1 \in B$. Since the sequence is decreasing, by replacing q_2 with $J'(q_2, r_2, q_2)$ we can replace r_2 with r . After doing this replacement we have

$$\begin{aligned} r &= J'(r, q_1, e_{j_1}(\bar{n}_1, r)), \\ r_2 &= J'(r, q_1, e_{j_1}(\bar{n}_1, r_2)) = J'(r, q_2, e_{j_2}(\bar{n}_2, r_2)), \\ t_1 &= J'(r, q_2, e_{j_2}(\bar{n}_2, s)) = J(t_1, \rho_1, r'_2), \text{ and} \\ s &= J(r_2, \rho_1, s'). \end{aligned}$$

We will analyze the subsequence r, r_2, t_1 .

Let $t = J'(r, q_1, e_{j_1}(\bar{n}_1, t_1))$. We will show that

$$\begin{aligned} r &= J'(r, q_1, e_{j_1}(\bar{n}_1, r)), & t &= J'(r, q_1, e_{j_1}(\bar{n}_1, t_1)) = J(t, q_1, r'), \\ t_1 &= J(t, q_1, t'_1), & \text{for } r' &= e_2(r, q_1, r) \text{ and } t'_1 = e_2(r, q_1, t_1). \end{aligned}$$

The only equalities that have not been shown already are $t = J(t, q_1, r')$ and $t_1 = J(t, q_1, t'_1)$. As usual, let $\mathbb{B} \leq \prod_{l \in L} \mathbb{C}_l$ be a subdirect representation of \mathbb{B} by subdirectly irreducible algebras. We will proceed componentwise.

We will begin by showing that $t = J(t, q_1, r')$. Since $J(t, q_1, r') \leq t$, by the flatness of \mathbb{C}_l , it will be sufficient to show that $t(l) \neq 0$ implies $J(t, q_1, r')(l) \neq 0$. Suppose that $t(l) \neq 0$. Since $t = J'(r, q_1, e_{j_1}(\bar{n}_1, t_1))$, either $t(l) = q_1(l)$ or $t(l) = \partial q_1(l)$. If $t(l) = q_1(l)$, then $J(t, q_1, r')(l) = (t \wedge q_1)(l) = t(l)$. Suppose now that $t(l) = \partial q_1(l)$, then $r(l) = t(l)$, since \mathbb{C}_l is flat and $t \leq r$. Therefore $r'(l) = e_2(r, q_1, r)(l) = r(l) = t(l)$, and so $J(t, q_1, r')(l) = t(l)$. Hence $J(t, q_1, r') = t$.

Next, we show that $t_1 = J(t, q_1, t'_1)$. Again, we will assume that $t(l) \neq 0$, since $t(l) = 0$ implies that $t_1(l) = 0$ and $J(t, q_1, t'_1)(l) = 0$. Since \mathbb{C}_l is flat, if $t(l) \neq 0$, then $t_1(l) = t(l)$ and $t(l) \in \{q_1(l), \partial q_1(l)\}$. If $t(l) = q_1(l)$. Then $J(t, q_1, t'_1)(l) = t(l) = t_1(l)$. If $t(l) = \partial q_1(l)$, then $t'_1(l) = e_2(r, q_1, t_1) = t_1(l)$, so $J(t, q_1, t'_1)(l) = t'_1(l) = t_1(l)$. Hence $J(t, q_1, t'_1) = t_1$.

We now have

$$\begin{aligned} r &= J'(r, q_1, e_{j_1}(\bar{n}_1, r)), & t &= J'(r, q_1, e_{j_1}(\bar{n}_1, t_1)) = J(t, q_1, r'), \\ t_1 &= J(t, q_1, t'_1) = J(t_1, \rho_1, r'_2), & s &= J(t_1, \rho_1, s'), \end{aligned}$$

where $r' = e_2(r, q_1, r)$, $t'_1 = e_2(r, q_1, t_1)$, $r'_2 = e_2(t_1, \rho_1, t_1)$, and $s' = e_2(r_2, \rho_1, s)$. Apply Lemma 4.11 to the sequence t, t_1, s (the part of the sequence in the range of J) to get an element $\rho \in B$ such that $t = J(t, \rho, t'')$ and $s = J(t, \rho, s'')$ for $t'' = e_2(t, \rho, t)$ and $s'' = e_2(t, \rho, s)$. Since $t \leq r$, if $r' = e_2(r, \rho, r)$ and $s' = e_2(r, \rho, s)$, we have $t = J(t, \rho, r')$ and $s = J(t, \rho, s')$. Finally, we now have

$$\begin{aligned} r &= J'(r, q_1, e_{j_1}(\bar{n}_1, r)), & t &= J'(r, q_1, e_{j_1}(\bar{n}_1, s)) = J(t, \rho, r') \\ s &= J(t, \rho, s'), & \text{for } r' &= e_2(r, \rho, r) \text{ and } s' = e_2(r, \rho, s), \end{aligned}$$

proving the lemma. \square

In light of the above lemma, define

$$(4.8) \quad \psi_{J'J}(w, x, y, z) = \exists t [\alpha(t, w, x, y, z) \wedge \beta(t, w, x, y, z)],$$

where

$$\alpha(t, w, x, y, z) = \bigvee_{i=0}^2 \exists \bar{n}, a [\psi_S(e_i(\bar{n}, w), e_i(\bar{n}, x), y, z) \\ \wedge w = J'(w, a, e_i(\bar{n}, w)) \wedge t = J'(w, a, e_i(\bar{n}, x))]$$

and

$$\beta(t, w, x, y, z) = \exists b [\psi_S(e_2(w, b, w), e_2(w, b, x), y, z) \\ \wedge t = J(t, b, e_2(w, b, w)) \wedge x = J(t, b, e_2(w, b, x))].$$

Recall that ψ_S was defined in (4.5). If $c, d \in e_i(\bar{n}, B)$ for some $i \in \{0, 1, 2\}$ and some $\bar{n} \in B^2 \cup B$ and $(r, s) \in \text{Cg}^{\mathbb{B}}(c, d)$ is witnessed by a chain satisfying the hypotheses of Lemma 4.12, then $(r, s) \in \text{Cg}^{\mathbb{B}}(c, d)$ if and only if $\mathbb{B} \models \psi_{J'J}(r, s, c, d)$.

At this point, we have the machinery necessary to change a general decreasing Maltsev chain into a longer chain whose associated polynomials all have J , J' , or S_j as the outermost operations, and then to collapse repeated occurrences of J and J' to either a single occurrence of J or the chain $J'-J$. In order to fully collapse the chain, we still need to address what happens when the chain has alternating J and J' operations.

Lemma 4.13. *Let $r, t, s \in B$ be such that $s \leq t \leq r$ and $(r, t), (t, s) \in \text{Cg}^{\mathbb{B}}(c, d)$ for some $c, d \in B$. Suppose that for constants $p_1, p_2, q_1, q_2 \in B$,*

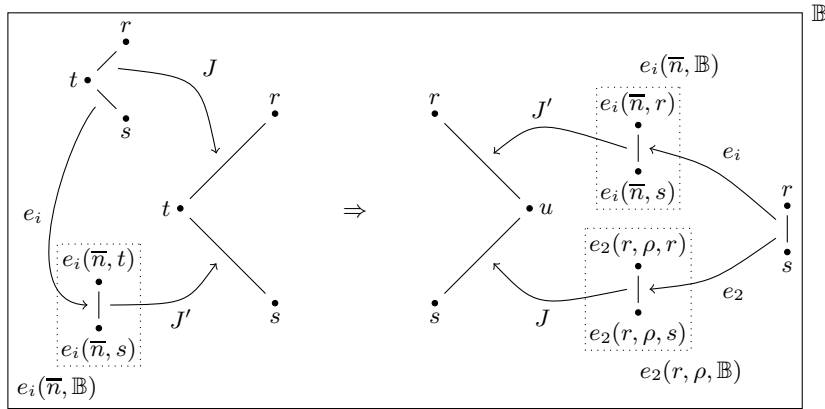
$$\begin{aligned} r &= J(p_1, q_1, r), & t &= J(p_1, q_1, t) = J'(p_2, q_2, t'), \\ s &= J'(p_2, q_2, s'), \end{aligned}$$

for $t' = e_i(\bar{n}, t)$ and $s' = e_i(\bar{n}, s)$ for some $i \in \{0, 1, 2\}$ and $\bar{n} \in B^2 \cup B$. Then there exist constants $\rho, u \in B$ such that

$$\begin{aligned} r &= J'(r, \rho, r') & u &= J'(r, \rho, s') = J(u, \rho, r'') \\ s &= J(u, \rho, s''), \end{aligned}$$

where $r' = e_i(\bar{n}, r)$, $r'' = e_2(r, \rho, r)$, and $s'' = e_2(r, \rho, s)$.

In the terminology of Definition 4.9, for every J - J' chain there is a J' - J chain with the same endpoints.



Proof. Since $s \leq t \leq r$, by replacing q_1 with $J(p_1, q_1, p_1)$ and q_2 with $J'(p_2, q_2, p_2)$, we can replace p_1 and p_2 with r . Thus,

$$\begin{aligned} r &= J(r, q_1, r) = (r \wedge \partial q_1 \wedge r) \vee (r \wedge q_1), \\ t &= J(r, q_1, t) = (r \wedge \partial q_1 \wedge t) \vee (r \wedge q_1) \\ &= J'(r, q_2, t') = (r \wedge q_2 \wedge t') \vee (r \wedge \partial q_2), \text{ and} \\ s &= J'(r, q_2, s') = (r \wedge q_2 \wedge s') \vee (r \wedge \partial q_2). \end{aligned}$$

Let $\rho = K(r, q_1, q_2)$ and $u = J'(r, \rho, s')$. Let $\mathbb{B} \leq \prod_{l \in L} \mathbb{C}_l$ be a subdirect representation of \mathbb{B} by subdirectly irreducible algebras. We will show that the equalities in the conclusion of the statement of the lemma hold componentwise.

We begin by showing that $r = J'(r, \rho, r')$. As usual, a table is the easiest way to organize the proof. Since $r(l) = 0$ implies $J'(r, \rho, r')(l) = 0$, assume that $r(l) \neq 0$.

r	$\rho = K(r, q_1, q_2)$	$J'(r, \rho, r')$	$r \neq t$
$q_1 = q_2$	$q_1 = q_2 = r$	$r \wedge r'$	N
$q_1 = \partial q_2$	$q_2 = \partial r$	r	N
$\partial q_1 = q_2$	$q_1 = \partial r$	r	Y
$\partial q_1 = \partial q_2$	$q_1 = \partial r$	r	N
$q_1 \notin \{q_2, \partial q_2\}$	0	0	N
$\partial q_1 \notin \{q_2, \partial q_2\}$	$q_1 = \partial r$	r	Y

The only possibly problematic cases are when $r(l) = q_1(l) = q_2(l)$ and when $r(l) = q_1(l) \notin \{q_2(l), \partial q_2(l)\}$. Suppose that $r(l) = q_1(l) = q_2(l)$. If $r(l) = q_1(l)$, then $r(l) = t(l)$, by the above equations, so $r(l) = t(l) = t'(l)$. Since $t'(l) \leq r'(l)$ (because $e_i(\bar{n}, -)$ is monotonic and $r \geq t$), it follows that $r'(l) = r(l)$, so $J(r, \rho, r')(l) = r(l)$ in this case. Next, suppose that $r(l) = q_1(l) \notin \{q_2(l), \partial q_2(l)\}$. If $r(l) = q_1(l)$, then $r(l) = t(l)$, but if $r(l) \notin \{q_2(l), \partial q_2(l)\}$ then $t(l) = 0$, contradicting our assumption that $r(l) \neq 0$. Therefore, in this case $J'(r, \rho, r')(l) = r(l)$ as well.

Next, we show that $u = J(u, \rho, r'')$. Since $J(u, \rho, r'') \leq u$ and each \mathbb{C}_l is flat, it will be sufficient to show that when $u(l) \neq 0$, $J(u, \rho, r'')(l) \neq 0$ as well. When $u(l) \neq 0$, since $u \leq r$, it must be that $u(l) = r(l)$. If $\rho(l) = r(l)$, then $J(u, \rho, r'')(l) = r(l) = u(l)$. Suppose now that $\rho(l) = \partial r(l)$. Then $r''(l) = e_2(r, \rho, r)(l) = r(l) = u(l)$, so $J(u, \rho, r'')(l) = r(l)$. Since $\rho(l) = r(l)$ or $\rho(l) = \partial r(l)$ for all $l \in L$, we have that $u = J(r, \rho, r'')$.

Finally, we show that $s = J(u, \rho, s'')$. First, suppose that $r(l) = s(l)$. Then $u(l) = r(l) = s(l)$, since $s \leq u \leq r$. If $\rho(l) = r(l)$, then $J(u, \rho, s'') = (u \wedge \rho)(l) = r(l) = s(l)$. If $\rho(l) = \partial r(l)$, then $s''(l) = e_2(r, \rho, s)(l) = s(l)$, so $J(u, \rho, s'') = (u \wedge \rho \wedge s'')(l) = s(l)$. Next, suppose that $r(l) = u(l)$ but $s(l) = 0$. If $\rho(l) = r(l)$, then $r(l) = q_1(l) = q_2(l)$, so $s(l) = J'(r, q_2, s')(l) = J'(r, \rho, s')(l) = u(l)$, contradicting $u(l) \neq 0$. If $\rho(l) = \partial r(l)$, then $s''(l) = e_2(r, \rho, s) = s(l)$, so $J(u, \rho, s'')(l) = s''(l) = s(l)$. Finally, suppose that $r(l) \neq 0$ and $u(l) = s(l) = 0$. If $\rho(l) = r(l)$, then $J(u, \rho, r'')(l) = u(l) = J(u, \rho, s'')(l)$, so $s(l) = J(u, \rho, s'')$. If $\rho(l) = \partial r(l)$, then $s''(l) = e_2(r, \rho, s) = s(l)$, so $J(u, \rho, s'')(l) = s''(l) = s(l)$. In all cases, we have $J(u, \rho, s'')(l) = s(l)$, so it must be that $J(u, \rho, s'') = s$, completing the proof. \square

Lemma 4.12 allows us to reduce a chain consisting of a string of J' operations to a J' - J chain. Since a chain of length 1 with a single J' operation cannot be defined

this way, let

$$(4.9) \quad \psi_{J'}(w, x, y, z) = \bigvee_{i=0}^2 \exists \bar{n} [\psi_S(e_i(\bar{n}, w), e_i(\bar{n}, x), y, z) \\ \wedge w = J'(a, b, e_i(\bar{n}, w)) \wedge x = J'(a, b, e_i(\bar{n}, x))]$$

(ψ_S was defined in (4.5)). Then if $c, d \in e_j(\bar{m}, B)$ for some $j \in \{0, 1, 2\}$ and some $\bar{m} \in B^2 \cup B$, and $(r, s) \in \text{Cg}^{\mathbb{B}}(c, d)$ is witnessed by $r \stackrel{J'}{\leq} s$, by Lemma 4.8, there must be $i \in \{0, 1, 2\}$ and $\bar{n} \in B^2 \cup B$ such that $r = J'(a, b, e_i(\bar{n}, r))$ and $s = J'(a, b, e_i(\bar{n}, s))$ for some $a, b \in B$. Therefore $\mathbb{B} \models \psi_{J'}(r, s, c, d)$ if and only if $(r, s) \in \text{Cg}^{\mathbb{B}}(c, d)$ and it is witnessed by a J' chain.

Since $e_i(\bar{n}, \mathbb{B})$ has that property that $a \in e_i(\bar{n}, B)$ and $b \leq a$ implies $b \in e_i(\bar{n}, B)$, for any decreasing Maltsev chain, if any one of the associated polynomials lies in the range of an S_i , then all subsequent polynomials lie in the range of S_i as well. Thus, every Maltsev chain must terminate in a (possibly length 0) S_i chain. Lemma 4.10 allows us to collapse such chains. Hence, to the already defined ψ_S , ψ_J , $\psi_{J'}$, and $\psi_{J'J}$ we add the following

$$(4.10) \quad \psi_{JS}(w, x, y, z) = \exists t [\psi_J(w, t, y, z) \wedge \psi_S(t, x, y, z)],$$

$$(4.11) \quad \psi_{J'S}(w, x, y, z) = \exists t [\psi_{J'}(w, t, y, z) \wedge \psi_S(t, x, y, z)], \text{ and}$$

$$(4.12) \quad \psi_{J'JS}(w, x, y, z) = \exists t [\psi_{J'J}(w, t, y, z) \wedge \psi_S(t, x, y, z)]$$

(see equations (4.5), (4.7), (4.9), and (4.8) for definitions of ψ_S , ψ_J , $\psi_{J'}$, and $\psi_{J'J}$, respectively).

Lemma 4.14. *Let $c, d \in e_i(\bar{m}, B)$ for some $i \in \{0, 1, 2\}$ and $\bar{m} \in B^2 \cup B$. If $(r, s) \in \text{Cg}^{\mathbb{B}}(c, d)$ is witnessed by a decreasing Maltsev sequence whose associated polynomials are generated by fundamental translations, then $(r, s) \in \text{Cg}^{\mathbb{B}}(c, d)$ is witnessed by one of the following chains:*

- (1) S_j for some $j \in \{0, 1, 2\}$ and $\mathbb{B} \models \psi_S(r, s, c, d)$,
- (2) J and $\mathbb{B} \models \psi_J(r, s, c, d)$,
- (3) J' and $\mathbb{B} \models \psi_{J'}(r, s, c, d)$,
- (4) $J'-J$ and $\mathbb{B} \models \psi_{J'J}(r, s, c, d)$,
- (5) $J-S_j$ for some $j \in \{0, 1, 2\}$ and $\mathbb{B} \models \psi_{JS}(r, s, c, d)$,
- (6) $J'-S_j$ for some $j \in \{0, 1, 2\}$ and $\mathbb{B} \models \psi_{J'S}(r, s, c, d)$, or
- (7) $J'-J-S_j$ for some $j \in \{0, 1, 2\}$ and $\mathbb{B} \models \psi_{J'JS}(r, s, c, d)$.

Proof. In all of the cases, that \mathbb{B} models the claimed first order formula follows from the definition of the formula and the conclusion of the appropriate lemmas: 4.10 for formulas whose subscript ends in S , 4.11 for formulas whose subscripts begins in J , and 4.12 and 4.8 for formulas whose subscript begin with J' .

Let $r = r_1, r_2, \dots, r_n = s$ be the decreasing Maltsev sequence witnessing $(r, s) \in \text{Cg}^{\mathbb{B}}(c, d)$ and let $\lambda_1(x), \dots, \lambda_{n-1}(x)$ be the polynomials generated by fundamental translations associated to it. From Lemma 4.8, without loss of generality we may assume that for each $k \in \{1, \dots, n-1\}$ one of the following holds

- (1) $\lambda_k(x) = S_{j_k}(\bar{m}_k, r_k, r_{k+1}, h_k(x))$ for some $j_k \in \{0, 1, 2\}$ and $\bar{m}_k \in B^2 \cup B$,
- (2) $\lambda_k(x) = J(r_k, q_k, h_k(x))$ for some $q_k \in B$, or
- (3) $\lambda_k(x) = J'(r_k, q_k, h_k(x))$ for some $q_k \in B$,

where the polynomials $h_k(x)$ are generated by fundamental translations and for each k either $\text{Range}(h_k) \subseteq \text{Range}(S_{j_k})$ for some $j_k \in \{0, 1, 2\}$ or $h_k(x)$ is 0-absorbing.

Since $e_{j_k}(\bar{n}, B)$ has the property that if $q \in e_{j_k}(\bar{n}, B)$ and $p \leq q$ then $p \in e_{j_k}(\bar{n}, B)$, if $\text{Range}(\lambda_k) \subseteq \text{Range}(S_{j_k})$, then

$$r_1 \xrightarrow{F_1} r_2 \xrightarrow{F_2} r_3 \cdots r_{k-1} \xrightarrow{F_{k-1}} r_k \xrightarrow{S_{j_k}} r_{k+1} \xrightarrow{S_{j_k}} r_{k+2} \cdots r_{n-1} \xrightarrow{S_{j_k}} r_n \quad F_i \in \{J, J'\}.$$

Lemma 4.10 can be applied to the pair $(r_k, s) \in \text{Cg}^{\mathbb{B}}(c, d)$ to collapse the end of the chain, and produce a new shorter chain of the form $F_1-F_2-\cdots-S_{j_k}$.

From Lemmas 4.11 and 4.12, subchains consisting of entirely J or J' can be converted to subchains consisting of a single J or $J'-J$, respectively:

$$\begin{aligned} J - J - \cdots - J &\Rightarrow J, \\ J' - J' - \cdots - J' &\Rightarrow J' - J \end{aligned}$$

Thus, we need only consider chains in which the J and J' are mixed. We will show that all such chains can be reduced to $J' - J$ chains. We have

$$\begin{aligned} J - J' - J' &\Rightarrow J - J' - J \Rightarrow J' - J - J \Rightarrow J' - J, \\ J' - J - J' &\Rightarrow J' - J' - J \Rightarrow J' - J - J \Rightarrow J' - J, \\ J' - J' - J &\Rightarrow J' - J - J \Rightarrow J' - J, \\ J' - J - J &\Rightarrow J' - J, \\ J - J' - J &\Rightarrow J' - J - J \Rightarrow J' - J, \text{ and} \\ J - J - J' &\Rightarrow J - J' \Rightarrow J' - J \end{aligned}$$

(using Lemmas 4.11, 4.12, and 4.13). It follows that all mixed chains of J and J' can be reduced to a $J'-J$ chain. The conclusion of the lemma follows. \square

Given the previous lemma, let

$$\begin{aligned} \psi_1(w, x, y, z) &= \psi_S(w, x, y, z) \vee \psi_J(w, x, y, z) \vee \psi_{J'}(w, x, y, z) \\ &\vee \psi_{J'J}(w, x, y, z) \vee \psi_{JS}(w, x, y, z) \vee \psi_{J'S}(w, x, y, z) \vee \psi_{J'JS}(w, x, y, z). \end{aligned}$$

All of the lemmas above required that $s \leq r$. Since \mathbb{B} is a semilattice, if $(r, s) \in \text{Cg}^{\mathbb{B}}(c, d)$, then there is an intermediate element, $t \leq r \wedge s$, such that $(r, t), (t, s) \in \text{Cg}^{\mathbb{B}}(c, d)$ and there are decreasing Maltsev chains connecting r to t and s to t (this will be proved in detail in Theorem 4.15). Therefore, define

$$(4.13) \quad \psi_2(w, x, y, z) = \exists t [\psi_1(w, t, y, z) \wedge \psi_1(x, t, y, z)].$$

Finally, we use the above lemmas to prove there is a congruence formula Γ_1 (defined in the theorem below) such that if $a, b \in B$ are distinguished by a polynomial of the form $e_i(\bar{n}, x)$ for some $i \in \{0, 1, 2\}$ and some \bar{n} , then $\text{Cg}^{\mathbb{B}}(a, b)$ has a subcongruence witnessed by $\Gamma_1(-, -, a, b)$ and that this subcongruence is defined by ψ_2 .

Theorem 4.15. *Let $a, b \in B$ and suppose that there is $i \in \{0, 1, 2\}$ and $\bar{m} \in B^2 \cup B$ such that $e_i(\bar{m}, a) \neq e_i(\bar{m}, b)$. Let*

$$\Gamma_1(w, x, y, z) = \bigvee_{j=0}^2 \exists \bar{n} \Gamma_0(w, x, e_j(\bar{n}, y), e_j(\bar{n}, z))$$

(Γ_0 was defined in (4.4)). *The congruence $\text{Cg}^{\mathbb{B}}(a, b)$ has a principal subcongruence witnessed by $\Gamma_1(-, -, a, b)$ and defined by ψ_2 . That is,*

$$\mathbb{B} \models \exists c, d [\Gamma_1(c, d, a, b) \wedge \Pi_{\psi_2}(c, d)].$$

Proof. From the definition of e_i (4.2) and Lemma 4.2, $e_i(\overline{m}, \mathbb{B})$ is congruence distributive and has definable principal subcongruences witnessed by Γ_0 and ψ_0 (defined in (4.4)). Therefore there are $c, d \in e_i(\overline{m}, B)$ such that

$$e_i(\overline{m}, \mathbb{B}) \models \Gamma_0(c, d, e_i(\overline{m}, a), e_i(\overline{m}, b)) \quad \text{and} \quad e_i(\overline{m}, \mathbb{B}) \models \Pi_{\psi_2}(c, d).$$

Since Γ_0 is a congruence formula, $\mathbb{B} \models \Gamma_1(c, d, a, b)$. It remains to be shown that $\mathbb{B} \models \Pi_{\psi_2}(c, d)$ (that is, that $\text{Cg}^{\mathbb{B}}(c, d)$ is defined in \mathbb{B} by ψ_2).

Let $r, s \in B$ and $(r, s) \in \text{Cg}^{\mathbb{B}}(c, d)$. To show that $\mathbb{B} \models \psi_2(r, s, c, d)$, by Lemma 4.14 we need only show that there are decreasing Maltsev sequences connecting r to some t and s to t and whose associated polynomials are generated by fundamental translations.

Let $r = r_1, \dots, r_n = s$ be a Maltsev sequence connecting r to s with associated polynomials $\lambda_1(x), \dots, \lambda_{n_1}(x)$ generated by fundamental translations. Let

$$t_i = \begin{cases} r_1 \wedge r_2 \wedge \dots \wedge r_i & \text{if } i \leq n, \\ t_{i-n} \wedge r_{i-n+1} \wedge \dots \wedge r_n & \text{if } n \leq i \leq 2n, \end{cases}$$

and

$$\mu_i(x) = \begin{cases} \lambda_i(x) \wedge t_i & \text{if } i < n, \\ \lambda_{i-n} \wedge t_{i+1} & \text{if } n < i \leq 2n. \end{cases}$$

Then the sequences $r = t_1, t_2, \dots, t_n$ and $s = t_{2n}, \dots, t_{n+1} = t_n$ are decreasing Maltsev sequences witnessed by the polynomials $\mu_i(x)$, which are generated by fundamental translations. Thus,

$$\mathbb{B} \models \psi_1(r, t_n, c, d) \wedge \psi_1(t_n, s, c, d),$$

and hence $\mathbb{B} \models \psi_2(r, s, c, d)$. From the definition of ψ_2 , it is a congruence formula, so if $\mathbb{B} \models \psi_2(u, v, c, d)$ then $(u, v) \in \text{Cg}^{\mathbb{B}}(c, d)$. Therefore $\mathbb{B} \models \Pi_{\psi_2}(c, d)$. \square

Having completed the argument for the case when $a, b \in B$ are distinguished by a polynomial of the form $e_i(\overline{m}, x)$ for some $i \in \{0, 1, 2\}$ and some $\overline{m} \in B^2 \cup B$, we move on to the case when a, b are distinguished by an operation from a sequential type SI. The next lemma is crucial for this case as well as the case where the coordinate is of machine type.

Lemma 4.16. *Let $c, d \in B$ be such that $d \leq c$ and $e_i(\overline{m}, c) = e_i(\overline{m}, d)$ for all $i \in \{0, 1, 2\}$ and all $\overline{m} \in B^2 \cup B$. Suppose that*

$$\begin{aligned} r &= f_1(c), & t &= f_1(d) = f_2(c), \\ s &= f_2(d), \end{aligned}$$

for some polynomials $f_1(x)$ and $f_2(x)$. Then $r = t$ or $t = s$.

Proof. Suppose that $t \neq s$. We will show that $r = t$. Let $\mathbb{B} \leq \prod_{l \in L} \mathbb{C}_l$ be a subdirect representation of \mathbb{B} by subdirectly irreducible algebras. Since each \mathbb{C}_l is flat and $r \geq t > s$, there is $k \in L$ such that $r(k) = t(k) \neq 0$, and $s(k) = 0$.

Claim: if $s(k) = 0$ then $d(k) = 0$.

Proof of claim: Suppose to the contrary that $d(k) \neq 0$ but $0 = s(k) = f_2(d)(k)$. Since \mathbb{C}_k is flat, $d(k) \neq 0$ implies that $d(k) = c(k)$, so

$$t(k) = f_2(c)(k) = f_2(d)(k) = s(k) = 0.$$

This contradicts our choosing k such that $t(k) \neq s(k) = 0$, and proves the claim.

By the above claim, we have that $d(k) = 0$, but since the only \mathbb{C}_l where $d(l) \neq c(l)$ are 0-absorbing, this implies that either $f_1(d)(k) = 0$, contradicting $t(k) = f_1(d)(k) \neq s(k) = 0$, or that $f_1(c)(l) = f_1(d)(l)$ for all l such that \mathbb{C}_l is 0-absorbing (i.e. $f_1(x)$ doesn't depend on x in the 0-absorbing \mathbb{C}_l). Since $c(l)$ and $d(l)$ can only differ when \mathbb{C}_l is 0-absorbing, this means that $f_1(c) = f_1(d)$, implying that $r = t$. \square

We already have a detailed description of the sequential and machine type SI's and their monoliths, given at the end of Section 3 and by Theorem 3.4. The small SI's break into 2 broad categories – those that satisfy $\exists \bar{n}[e_i(\bar{n}, x) \approx x]$ and those that do not (in which case they satisfy $e_i(\bar{y}, x) \approx 0$ for all $i \in \{0, 1, 2\}$). Theorem 4.15 handles the algebras that satisfy $\exists \bar{n}[e_i(\bar{n}, x) \approx x]$, and we will prove that there are only three different isomorphism types for small SI algebras satisfying $e_i(\bar{y}, x) \approx 0$ for all $i \in \{0, 1, 2\}$.

Two of the small SI's that do not satisfy $\exists \bar{n}[e_i(\bar{n}, x) \approx x]$ are subalgebras of $\mathbb{A}'(\mathcal{T})$, and the remaining one is the 4-element quotient

$$(4.14) \quad \mathbb{W} = \langle H, C \rangle / \text{Cg}(M_1^0, 0) = \{0, H, C, D, M_1^0\} / \text{Cg}(M_1^0, 0)$$

(this will be proved in Lemma 4.17). The fundamental operations of $\mathbb{A}'(\mathcal{T})$ are all identically 0 in \mathbb{W} except for \wedge , which makes $\langle W; \wedge \rangle$ a flat semilattice, and the following:

$$\begin{aligned} x \cdot y &= 0 && \text{except for } H \cdot C = D, \\ T(w, x, y, z) &= 0 && \text{except for } T(H, C, H, C) = D, \\ J(x, y, z) &= x \wedge y, \text{ and } J'(x, y, z) = K(x, y, z) = x \wedge y \wedge z. \end{aligned}$$

Lemma 4.17. *Let $\mathbb{B} \in \mathbf{HS}(\mathbb{A}'(\mathcal{T}))$ be nontrivial and subdirectly irreducible and such that $\mathbb{B} \models e_i(\bar{y}, x) \approx 0$ for all $i \in \{0, 1, 2\}$. Then \mathbb{B} is isomorphic to the two element subalgebra $\{0, C\} \leq \mathbb{A}'(\mathcal{T})$, the three element subalgebra $\{0, H, M_1^0\} \leq \mathbb{A}'(\mathcal{T})$, or to the 4-element quotient \mathbb{W} .*

Proof. We will first consider subalgebras. Suppose that $\mathbb{B} \leq \mathbb{A}'(\mathcal{T})$ is SI. Since $\mathbb{B} \models S_i(\bar{n}, x, x, x) \approx 0$ for all i , we have $(\{1, 2\} \cup V_0) \cap B = \emptyset$ and $\{x, \partial x\} \not\subseteq B$ for $x \in W \cup V$ (i.e. the “bar-able” elements of $\mathbb{A}'(\mathcal{T})$). All fundamental operations are identically 0 except for \wedge and

$$(4.15) \quad \begin{aligned} I(x) &= 0 \text{ except for } I(H) = M_1^0, \\ x \cdot y &= 0 \text{ except for } H \cdot C = D \text{ and } H \cdot \partial C = \partial D, \\ J(x, y, z) &= x \wedge y, & J'(x, y, z) &= x \wedge y \wedge z, \\ T(w, x, y, z) &= (w \wedge y) \cdot (x \wedge z). \end{aligned}$$

There are two cases depending upon whether or not H is an element of B . First, suppose that $H \notin B$. Then $x \cdot y = T(w, x, y, z) = I(x) = 0$, so \mathbb{B} is a flat semilattice. It follows that if $x, y \in B$ are distinct and nonzero, then $\text{Cg}^{\mathbb{B}}(x, 0)$ and $\text{Cg}^{\mathbb{B}}(y, 0)$ cover $\mathbf{0}$ in $\text{Con}(\mathbb{B})$, and hence \mathbb{B} is not subdirectly irreducible. Therefore $B = \{0, x\}$, and so \mathbb{B} is isomorphic to the subalgebra $\{0, C\}$.

Next, suppose that $H \in B$. Then $I(H) = M_1^0 \in B$ as well. We will show that $\text{Cg}^{\mathbb{B}}(M_1^0, 0)$ is the monolith of \mathbb{B} . Suppose that $\text{Cg}^{\mathbb{B}}(M_1^0, 0)$ is not the monolith. Then there is some polynomial $f(x)$ such that $f(M_1^0) \notin \{M_1^0, 0\}$, but if $F(x)$ is a

fundamental translation of \mathbb{B} , then $F(M_1^0) = M_1^0$ or $F(M_1^0) = 0$ (see the description of the fundamental operations above). It follows that for any polynomial $f(x)$ we have $f(M_1^0) \in \{M_1^0, 0\}$, contradicting our assumption. Hence $\text{Cg}^{\mathbb{B}}(M_1^0, 0)$ is the monolith of \mathbb{B} .

We will now show that $B = \{0, H, M_1^0\}$. Suppose that $x \in B \setminus \{0, H, M_1^0\}$. If $x = C$, then $H \cdot C = D$, so $D \in B$ as well. An argument similar to the previous paragraph will show that $\text{Cg}^{\mathbb{B}}(D, 0)$ covers $\mathbf{0}$. Likewise, if $x = \partial C$, then $\text{Cg}^{\mathbb{B}}(\partial D, 0)$ covers $\mathbf{0}$. Both of these cases contradict the subdirect irreducibility of \mathbb{B} . If $x \notin \{C, \partial C\}$ then $\text{Cg}^{\mathbb{B}}(x, 0)$ also covers $\mathbf{0}$, again contradicting \mathbb{B} being subdirectly irreducible. Therefore it must be that $B \setminus \{0, H, M_1^0\} = \emptyset$. It follows that the only subdirectly irreducible subalgebras of $\mathbb{A}'(\mathcal{T})$ are isomorphic to either $\{0, C\}$ or $\{0, H, M_1^0\}$.

Now we examine the case when \mathbb{B} is a quotient of a subalgebra of $\mathbb{A}'(\mathcal{T})$. Suppose that $\mathbb{B} = \mathbb{B}_1/\theta \in \mathbf{HS}(\mathbb{A}'(\mathcal{T}))$ is subdirectly irreducible. In the quotient \mathbb{B} , the equations (4.15) hold by the same argument appearing at the start of the proof. Since $\mathbb{A}'(\mathcal{T})$ is a flat semilattice, the only possibly nontrivial class of θ is the one containing 0. If $H \notin B_1$ or $(H, 0) \in \theta$, then \mathbb{B} is a subdirectly irreducible flat semilattice (i.e. all the operations are 0 except for \wedge), so \mathbb{B} must be isomorphic to the 2-element subalgebra $\{0, C\}$ of $\mathbb{A}'(\mathcal{T})$.

Suppose now that $H \in B_1$ and $(H, 0) \notin \theta$. There are three cases to consider:

- $(M_1^0, 0) \notin \theta$,
- $(M_1^0, 0) \in \theta$ and $[C \notin B_1 \text{ or } (C, 0) \in \theta]$, or
- $(M_1^0, 0) \in \theta$ and $[C \in B_1 \text{ and } (C, 0) \notin \theta]$.

If $(M_1^0, 0) \notin \theta$ then $\text{Cg}^{\mathbb{B}}(M_1^0, 0)$ is the monolith of \mathbb{B} , so by the last paragraph \mathbb{B} is isomorphic to the 3-element subalgebra $\{0, H, M_1^0\}$ of $\mathbb{A}'(\mathcal{T})$. If instead $(M_1^0, 0) \in \theta$ and $[C \notin B_1 \text{ or } (C, 0) \in \theta]$, then $\text{Cg}^{\mathbb{B}}(H, 0)$ must cover $\mathbf{0}$, so \mathbb{B} is isomorphic to the 2-element subalgebra $\{0, C\}$. Suppose now that $(M_1^0, 0) \in \theta$ and $[C \in B_1 \text{ and } (C, 0) \notin \theta]$. If $(D, 0) \in \theta$, then both $\text{Cg}^{\mathbb{B}}(H, 0)$ and $\text{Cg}^{\mathbb{B}}(C, 0)$ cover $\mathbf{0}$, contradicting the subdirect irreducibility. If $(D, 0) \notin \theta$, then $\text{Cg}^{\mathbb{B}}(D, 0)$ covers $\mathbf{0}$. In this case, an argument similar to the case when $B \in \mathbf{S}(\mathbb{A}'(\mathcal{T}))$ shows that $x \notin B_1$ or $(x, 0) \in \theta$ for all $x \in B_1 \setminus \{0, H, C, D\}$. In this case, \mathbb{B} is isomorphic to the algebra \mathbb{W} described in (4.14) above. \square

Lemma 4.18. *Let $\mathbb{B} \leq \text{prod}_{l \in L} \mathbb{C}_l$ be a subdirect representation of \mathbb{B} by subdirectly irreducible algebras, and suppose that $c, d \in B$ are such that*

- (1) $d \leq c$,
- (2) $e_i(\bar{n}, c) = e_i(\bar{n}, d)$ for all $i \in \{0, 1, 2\}$ and all $\bar{n} \in B^2 \cup B$, and
- (3) For each $l \in L$, $\text{Cg}^{\mathbb{C}_l}(c(l), d(l))$ lies in the monolith of \mathbb{C}_l .

If $g(x)$ is a polynomial of \mathbb{B} generated by fundamental translations such that $g(c) \neq g(d)$, then there is some $\rho \in B$ and some polynomial $g'(x) = J'(g(c), \rho, F(x))$ where

$$F(x) \in \{id(x)\} \cup \{F_1(a_1, b_1, F_2(a_2, b_2, \dots F_n(a_n, b_n, x) \dots)) \mid n \in \mathbb{N}, F_i \in \mathcal{L} \cup \mathcal{R}, \text{ and } p_i, q_i \in B\}$$

such that

$$g(d) = g'(d) = J'(g(c), \rho, F(d)) \quad \text{and} \quad g(c) = g'(c) = J'(g(c), \rho, F(c)).$$

Proof. For convenience, let $r = g(c)$ and $s = g(d)$ and

$$\mathcal{C} = \{\text{id}(x)\} \cup \{F_1(p_1, q_1, F_2(p_2, q_2, \dots F_n(p_n, q_n, x) \dots)) \mid n \in \mathbb{N}, F_i \in \mathcal{L} \cup \mathcal{R}, \text{ and } p_i, q_i \in B\}.$$

The proof will be broken down into two smaller claims. We will first show that if $g(x) = G(f(x))$ for a fundamental translation $G(x)$, then $g'(x) = J'(r, \rho, F(f(x)))$ satisfies $g'(c) = g(c)$ and $g'(d) = g(d)$ for some $\rho \in B$ and some $F(x) \in \mathcal{C}$.

Applying the above claim to the $f(x)$ in $J'(r, \rho, F(f(x)))$, we have that $g'(x) = J'(r, \rho, F(J'(r, \rho_1, E_1(f_1(x))))$ where $E_1 \in \mathcal{C}$ satisfies $g'(c) = r$ and $g'(d) = s$. Therefore, the second part of the proof will demonstrate that there is some $g''(x) = J'(r, \rho_2, E_2(f_1(x)))$ such that $g''(c) = g'(c) = r$ and $g''(d) = g'(d) = s$. Applying these two claims repeatedly to a general polynomial will yield the conclusion of the lemma.

Suppose first that $g(x) = G(f(x))$ for a fundamental translation $G(x)$. We will produce ρ and $F(x) \in \mathcal{C}$ such that $g'(x) = J'(r, \rho, F(f(x)))$ satisfies $g'(c) = g(c)$ and $g'(d) = g(d)$. The proof shall be by cases, depending on which particular fundamental operation G is a translation of. As usual, we shall proceed componentwise. The only possible $l \in L$ with $c(l) \neq d(l)$ are such that $\mathbb{C}_l \models e_i(\bar{n}, x) \approx 0$, by the second hypothesis and the description of SI algebras in Section 3. Therefore by Lemma 4.17 and from the hypotheses, the only fundamental translations that do not collapse $\text{Cg}^{\mathbb{B}}(c, d)$ are translations of the operations $\wedge, J, J', K, E, U_E^1$, and U_E^2 , where $E \in \mathcal{L} \cup \mathcal{R}$. In all cases except for operations from $\mathcal{L} \cup \mathcal{R}$, we will take $F = \text{id}$.

Before beginning the cases, note that if $g(x) = G(f(x)) \leq f(x)$, then since \mathbb{C}_l is flat either $r(l) = f(c)(l)$ or $r(l) = 0$, and likewise for $s(l)$. The polynomial $g'(x) = J'(r, r, f(x)) = r \wedge f(x)$ therefore has $g'(c) = r(l)$ and $g'(d) = s(l)$. Thus, in cases where $g(x) \leq f(x)$, taking $\rho = r$ and $F = \text{id}$ is sufficient.

Case \wedge : If $g(x) = u \wedge f(x)$, then $g(x) \leq f(x)$, so by the above remarks, take $\rho = r$.

Case J : If $g(x) = J(f(x), u, v)$, then $g(x) \leq f(x)$, so by the remarks at the start of the cases let $\rho = r$. If $g(x) = J(u, f(x), v)$, then let $\rho = K(r, f(c), r)$. Since many of the later cases are similar to this, we will carefully prove that $g'(x) = J'(r, \rho, f(x))$ satisfies $g'(c) = r$ and $g'(d) = s$. We have

$$g(x) = J(u, f(x), v) = (u \wedge f(x)) \vee (u \wedge \partial f(x) \wedge v).$$

This yields the following table (assume that $r(l) \neq 0$, since $g'(x)(l) = 0$ in that case).

r	$\rho = K(r, f(c), r)$	$g'(c)$	$g'(d)$
$f(c)$	r	$r \wedge f(c) = r$	$r \wedge f(d) = s$
$\partial f(c)$	$f(c) = \partial r$	r	r

The only problematic case is when $r(l) = \partial f(c)(l)$, but in this case we have that $s(l) = \partial f(d)(l)$, so $r(l) = e_2(r, f(c), r)(l)$ and $s(l) = e_2(r, f(c), s)(l)$, contradicting hypothesis (2) in the statement of the lemma. It follows that $g'(c) = r$ and $g'(d) = s$.

If $g(x) = J(u, v, f(x))$, then $g(c)(l)$ and $g(d)(l)$ agree whenever $u(l) = v(l)$ and can only possibly differ when $u(l) = \partial v(l)$. Hence, if $g(c) \neq g(d)$, then $e_2(u, v, g(c)) \neq e_2(u, v, g(d))$, contradicting the hypothesis that $e_i(\bar{n}, c) = e_i(\bar{n}, d)$ for all $i \in \{0, 1, 2\}$ and $\bar{n} \in B^2 \cup B$.

Case J' : If $g(x) = J'(f(x), u, v)$, then $g(x) \leq f(x)$, so by the remarks at the start of the cases let $\rho = r$. If $g(x) = J'(u, f(x), v)$, then let $\rho = K(r, f(c), r)$. An argument similar to the one in Case J will work.

Case K : If $g(x) = K(f(x), u, v)$, then let $\rho = K(r, f(c), r)$. If we have $g(x) = K(u, f(x), v)$, then let $\rho = K(r, u, r)$. If $g(x) = K(u, v, f(x))$, then let $\rho = K(r, u, r)$. Arguments similar to the one in Case J will work.

Case $E \in \mathcal{L} \cup \mathcal{R}$: $c(l)$ and $d(l)$ can only differ when \mathbb{C}_l is of sequential or machine type. In machine type SI's, by the third hypothesis, $c(l)$ must be a Turing machine configuration (see the description of SI's in Section 3). Since there are no non-constant polynomials that map a machine configuration to an sequential element (an element of Σ_N , using the notation from the description of the SI's in Section 3, it must be that $E(f(c), u, v)(l) = E(u, f(c), v)(l) = 0$ when \mathbb{C}_l is of machine type. Since $E(x, y, z) \approx 0$ in sequential SI's, we have that $E(f(c), u, v) = E(f(d), u, v)$ and $E(u, f(c), v) = E(u, f(d), v)$. Therefore we need only examine $g(x) = E(u, v, f(x))$. If $g(x) = E(u, v, f(x))$, then $g'(x) = J'(r, r, E(u, v, f(x))) = r \wedge E(u, v, f(x))$ clearly satisfies $g(c) = g'(c)$ and $g(d) = g'(d)$.

Case U_E^i for $E \in \mathcal{L} \cup \mathcal{R}$ and $i \in \{1, 2\}$: This case is quite similar to the previous one. Use the fact that $c(l)$ and $d(l)$ only differ on sequential and machine type \mathbb{C}_l , that $U_E^i(w, x, y, z) \approx 0$ on sequential SI's, and that

$$U_E^i(u, v, w, x) = 0 \quad \text{except for} \quad U_E^1(u, v, v, x) = E(u, v, x) = U_E^2(u, u, v, x)$$

in the machine type SI's.

Next, we will show that if $g(x) = J'(u, v, F(J'(p, q, E(f(x)))))$ for constants $u, v, p, q \in B$ and $F(x), E(x) \in \mathcal{C}$ then there is some ρ such that

$$g'(x) = J'(r, \rho, E_1(f(x)))$$

has $g'(c) = g(c) = r$ and $g'(d) = g(d) = s$. This is broken into two subclaims.

Claim: if $g(x) = G(J'(p, q, f(x)))$ for $G \in \mathcal{L} \cup \mathcal{R}$ then there is $\rho \in B$ such that $g'(x) = J'(r, \rho, G(f(x)))$ has $g'(c) = g(c) = r$ and $g'(d) = g(d) = s$.

Proof of claim: Let $\rho = G(q)$. We have

$$g(x) = G(J'(p, q, f(x))) = G((p \wedge \partial q) \vee (p \wedge q \wedge f(x))),$$

and $G(\partial x) = \partial G(x)$. Therefore $g'(x) = J'(r, G(q), G(f(x)))$ agrees with $g(x)$ on the set $\{c, d\}$, as claimed.

Claim: if $g(x) = J'(u, v, J'(p, q, f(x)))$ then there is some $\rho \in B$ such that $g'(x) = J'(r, \rho, f(x))$ has $g'(c) = g(c) = r$ and $g'(d) = g(d) = s$.

Proof of claim: Let $\rho = K(r, v, q)$. We have

$$\begin{aligned} g(x) &= J'(u, v, J'(p, q, f(x))) \\ &= (u \wedge v \wedge p \wedge q \wedge f(x)) \vee (u \wedge v \wedge p \wedge \partial q) \vee (u \wedge \partial v). \end{aligned}$$

This gives us the following table of cases (as usual, assume that $r(l) \neq 0$ since $g(x)(l) = 0$ in this case).

r	$\rho = K(r, v, q)$	$g'(c)$	$g'(d)$	$r \neq s$
$v = q$	r	$r \wedge f(c)$	$r \wedge f(d)$	Y
$v = \partial q$	$q = \partial r$	r	r	N
∂v	$v = \partial r$	r	r	N

In the case where $v(l) = q(l)$ we also have that $r(l) = f(c)(l)$ and $s(l) = f(d)(l)$, so the table above indicates that $g'(c) = r$ and $g'(d) = s$, as claimed.

These two claims show that any polynomial of the form $J'(u, v, F(f(x)))$ for $F(x) \in \mathcal{C}$ can be transformed into $J'(p, q, E(x))$ for $E(x) \in \mathcal{C}$ in such a way that the image of the pair (c, d) is preserved. Combined with the first claim in this proof, the conclusion of the lemma follows. \square

If $a, b \in B$ differ at a coordinate that is of sequential type, then the next lemma proves that there is some polynomial that maps (a, b) coordinatewise into the monoliths of the \mathbb{C}_l (the subdirect factors of \mathbb{B}) and does not collapse $\text{Cg}^{\mathbb{B}}(a, b)$.

Lemma 4.19. *There is a finite set of terms P depending only on $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ and such that if $a, b \in B$ are distinct, $e_i(\bar{n}, a) = e_i(\bar{n}, b)$ for all $i \in \{0, 1, 2\}$ and all $\bar{n} \in B^2 \cup B$, and there is $p \in B$ with $p \cdot a \neq p \cdot b$ or $a \cdot p \neq b \cdot p$, then there is $t(\bar{y}, x) \in P$ and $\bar{m} \in B^n$ with the property that if $c = t(\bar{m}, a)$ and $d = t(\bar{m}, b)$ then*

- $c \neq d$,
- $x \cdot c = x \cdot d$,
- $c \cdot x = d \cdot x$,
- $I(c) = I(d)$,
- $F(x, y, c) = F(x, y, d)$,
- $F(x, c, y) = F(x, d, y)$, and
- $F(c, x, y) = F(d, x, y)$

for $F \in \mathcal{L} \cup \mathcal{R}$ and for all $x, y \in B$.

Proof. Let $\mathbb{B} \leq \prod_{l \in L} \mathbb{C}_l$ be a subdirect representation of \mathbb{B} by subdirectly irreducible algebras. The subdirectly irreducible algebras in $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ can be divided into two groups: either $\mathbb{C}_l \models e_i(\bar{n}, x) \approx x$ for some $i \in \{0, 1, 2\}$ and some $\bar{n} \in C_l^2 \cup C_l$ or $\mathbb{C}_l \models e_i(\bar{n}, x) \approx 0$ for all $i \in \{0, 1, 2\}$. Since $e_i(\bar{n}, a) = e_i(\bar{n}, b)$, the projections $a(l)$ and $b(l)$ must agree on all factors that satisfy $\mathbb{C}_l \models e_i(\bar{n}, x) \approx x$ for some i and some \bar{n} , and can only possibly disagree on factors satisfying $\mathbb{C}_l \models e_i(\bar{n}, x) \approx 0$ for all i .

Claim: There is a finite number $N \in \mathbb{N}$ such that for all $l \in L$, if $\mathbb{C}_l \models e_1(n, x) \approx 0$ then

$$\mathbb{C}_l \models x_1 \cdot x_2 \cdots x_{N-1} \cdot x_N \approx 0.$$

Proof of claim: First recall that since \mathcal{T} halts, there are only finitely many subdirectly irreducible algebras, all finite. Therefore if \mathbb{C}_l does not model the identity in the claim, there must exist nonzero elements $r, s \in C_l$ such that $r \cdots r \cdot s = s$. Considering \mathbb{C}_l as a quotient of a product of subalgebras of $\mathbb{A}'(\mathcal{T})$, this means that there is some coordinate of the preimages (under the quotient map) of r and s such that $(r(i), s(i)) \in \{(1, C), (2, D)\}$. Therefore $e_1(r, s) \neq 0$, contradicting our assumption that $\mathbb{C}_l \models e_1(n, x) \approx 0$. Let S be a finite set containing a representative of each isomorphism type of the subdirectly irreducible algebras of $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$, and for $\mathbb{C} \in S$, let $n_{\mathbb{C}} \in \mathbb{N}$ be minimal such that $\mathbb{C} \models x_1 \cdots x_{n_{\mathbb{C}}} \approx 0$. Taking $N = \max\{n_{\mathbb{C}} \mid \mathbb{C} \in S\}$ completes the proof of the claim.

Since the product (\cdot) associates to the left, every polynomial of the form $f(x) = y_1 \cdots y_m \cdot x \cdot y_{m+1} \cdots y_M$ can be rewritten as $f(x) = y_1 \cdots y_m \cdot x \cdot z$, where $z = y_{m+1} \cdots y_M$. Let

$$P = \{f(y_1, \dots, y_M, x) = y_1 \cdots y_M \cdot x, \\ g(y_1, \dots, y_M, x) = y_1 \cdots y_{M-1} \cdot x \cdot y_M \mid 0 \leq M < N\}.$$

Thus, there is a term $t(\overline{y}, x) \in P_1$ and constants $\overline{m} \in B^n$ such that $t(\overline{m}, a) \neq t(\overline{m}, b)$ and $x \cdot t(\overline{m}, a) = x \cdot t(\overline{m}, b)$ and $t(\overline{m}, a) \cdot x = t(\overline{m}, b) \cdot x$ for all $x \in B$. Furthermore, since $p \cdot a \neq p \cdot b$ or $a \cdot p \neq b \cdot p$, the term t is not the identity. Thus $t(\overline{m}, x)(l) \approx 0$ when \mathbb{C}_l is of machine type (recall that machine type \mathbb{C}_l model $x \cdot y \approx 0$). Thus for all $x, y, z \in B$ and all $F \in \mathcal{L} \cup \mathcal{R}$,

$$F(t(\overline{m}, z), x, y) = F(x, t(\overline{m}, z), y) = F(x, y, t(\overline{m}, z)),$$

and $I(t(\overline{m}, x)) = 0$. \square

Let the set P be as in Lemma 4.19 and define

$$(4.16) \quad \Gamma_{(\cdot)}(w, x, y, z) = \bigvee_{t \in P \cup \{\text{id}(x)\}} \exists \overline{n} [w = t(\overline{n}, y) \wedge x = t(\overline{n}, z)].$$

Given Lemmas 4.18 and 4.16, define

$$(4.17) \quad \psi_{(\cdot)}(w, x, y, z) = \exists t [w = J'(w, t, y) \wedge x = J'(w, t, z)].$$

If $c, d \in B$ satisfy the conclusion of Lemma 4.19, then c, d also satisfy the hypotheses of Lemma 4.18. In this case, $(r, s) \in \text{Cg}^{\mathbb{B}}(c, d)$ if and only if $\mathbb{B} \models \psi_{(\cdot)}(r, s, c, d)$. Since we will be employing a strategy similar to the proof of Theorem 4.15, where a general Maltsev sequence is divided into 2 strictly decreasing sequences, let

$$(4.18) \quad \psi_3(w, x, y, z) = \exists t [\psi_{(\cdot)}(w, t, y, z) \wedge \psi_{(\cdot)}(x, t, y, z)].$$

Theorem 4.20. *Let $a, b \in B$ be distinct and such that $b \leq a$ and $e_i(\overline{n}, a) = e_i(\overline{n}, b)$ for all $i \in \{0, 1, 2\}$ and all $\overline{n} \in B^2 \cup B$. If one of the following*

- (1) *there is $p \in B$ such that $p \cdot a \neq p \cdot b$ or $a \cdot p \neq b \cdot p$, or*
- (2) *for all $u, v \in B$ and $F \in \mathcal{L} \cup \mathcal{R}$ each of the translations $x \cdot u$, $u \cdot x$, $I(x)$, $F(u, v, x)$, $F(u, x, v)$, and $F(x, u, v)$ agree for $x \in \{a, b\}$*

holds, then the congruence $\text{Cg}^{\mathbb{B}}(a, b)$ has a principal subcongruence witnessed by the formula $\Gamma_{(\cdot)}(-, -, a, b)$ and defined by the formula ψ_3 :

$$\mathbb{B} \models \exists c, d [c \neq d \wedge \Gamma_{(\cdot)}(c, d, a, b) \wedge \Pi_{\psi_3}(c, d)].$$

Proof. In the first case, where there is $p \in B$ such that $p \cdot a \neq p \cdot b$ or $a \cdot p \neq b \cdot p$, from Lemma 4.19, the pair (a, b) differs at a coordinate that is of sequential type, and $(a, b)(l)$ lies outside of the monolith of some sequential \mathbb{C}_l . In this case, find $t \in P$ and constants \overline{m} such that if $c = t(\overline{m}, a)$ and $d = t(\overline{m}, b)$ then

- $c \neq d$,
- $x \cdot c = x \cdot d$,
- $I(c) = I(d)$
- $c \cdot x = d \cdot x$,
- $F(x, y, c) = F(x, y, d)$,
- $F(x, c, y) = F(x, d, y)$, and
- $F(c, x, y) = F(d, x, y)$.

In the second case, where for all $u, v \in B$ and all $F \in \mathcal{L} \cup \mathcal{R}$ each of the translations $x \cdot u$, $u \cdot x$, $I(x)$, $F(u, v, x)$, $F(u, x, v)$, and $F(x, u, v)$ agree for $x \in \{a, b\}$, the pair (a, b) differ at a coordinate that is of sequential type, but $(a, b)(l)$ lies in the monolith of each sequential \mathbb{C}_l . In this case, let $c = a$ and $d = b$. In both of these cases, $\mathbb{B} \models \Gamma_{(\cdot)}(c, d, a, b)$ and c and d satisfy the hypotheses of Lemma 4.18.

Since $a \geq b$ and the operations of \mathbb{B} are monotonic, $c \geq d$. Suppose now that $(r, s) \in \text{Cg}^{\mathbb{B}}(c, d)$. Then using the same argument as in the proof of Theorem 4.15,

there are decreasing Maltsev chains $r = r_1, \dots, r_m = t$ and $s = s_1, \dots, s_n = t$ with associated polynomials generated by fundamental translations. Using first Lemma 4.18 and the description of c and d in the preceding paragraph, and then applying Lemma 4.16, we have that there are constants ρ and ρ' such that

$$\begin{aligned} r &= J'(r, \rho, c), & t &= J'(r, \rho, d) = J'(s, \rho', d), \\ s &= J'(s, \rho', c). \end{aligned}$$

Hence $\mathbb{B} \models \psi_{(\cdot)}(r, t, c, d) \wedge \psi_{(\cdot)}(s, t, c, d)$, completing the proof. \square

Next, we move on to analyzing the case where $a, b \in B$ differ at a coordinate that is of machine type. We will employ a strategy similar to the sequential case, and produce from (a, b) a pair (c, d) such that $(c, d)(l)$ lies in the monolith of \mathbb{C}_l for each $l \in L$.

Lemma 4.21. *There are finite sets of terms S and T depending only on $V(\mathbb{A}'(\mathcal{T}))$ such that if $a, b \in \mathbb{B}$ are distinct with $b \leq a$, $e_i(\bar{n}, a) = e_i(\bar{n}, b)$ for all $i \in \{0, 1, 2\}$ and all $\bar{n} \in B^2 \cup B$, and for some $F \in \mathcal{L} \cup \mathcal{R}$ and some $u, v \in B$ one of the following*

- (1) $F(u, v, a) \neq F(u, v, b)$, or
- (2) $F(u, v, I(a)) \neq F(u, v, I(b))$, or
- (3) (a) $I(a) = I(b)$,
 (b) $u \cdot a = u \cdot b$ and $a \cdot u = b \cdot u$, and
 (c) $F(u, v, a) = F(u, v, b)$,

holds, then there is $t(\bar{y}, x) \in S$ and constants $\bar{m} \in B$ such that if $c = t(\bar{m}, a)$ and $d = t(\bar{m}, b)$ then for any $n \in \mathbb{N}$ and $F_1, \dots, F_n \in \mathcal{L} \cup \mathcal{R}$ and any $a_1, \dots, a_{2n} \in B$ there is $G(\bar{z}, x) \in T$ and $b_1, \dots, b_m \in B$ such that

$$\begin{aligned} F_1(a_1, a_2, F_2(a_3, a_4, \dots, F_n(a_{2n-1}, a_{2n}, c) \dots)) &= G(b_1, \dots, b_m, c) & \text{and} \\ F_1(a_1, a_2, F_2(a_3, a_4, \dots, F_n(a_{2n-1}, a_{2n}, d) \dots)) &= G(b_1, \dots, b_m, d). \end{aligned}$$

Proof. If \mathbb{M} is a machine type SI, then using the notation from the discussion of large SI's in Section 3, the monolith of \mathbb{M} is $\text{Cg}^{\mathbb{M}}(\mathcal{P}, 0)$, and there are two possibilities for its structure: either

- $\mathcal{T}(\mathcal{P}) = 0$, in which case the only nontrivial class of the monolith is $\{\mathcal{P}, 0\}$, or
- there is $n \in \mathbb{N}$ such that $\mathcal{T}^n(\mathcal{P}) = \mathcal{T}(\mathcal{T}(\dots \mathcal{T}(\mathcal{P}) \dots)) = \mathcal{P}$ (that is, the Turing machine enters a non-terminating loop), in which case the only nontrivial class of the monolith is $\{\mathcal{P}, \mathcal{T}(\mathcal{P}), \dots, \mathcal{T}^{n-1}(\mathcal{P}), 0\}$.

Let K be an index set and $(\mathbb{C}_k)_{k \in K}$ a family of machine type SI's in $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$. Since $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ is residually finite and from the discussion of machine type SI's in Section 3, there is a finite set of terms S depending only on $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ such that for all $\mathbb{C} \leq \prod_{k \in K} \mathbb{C}_k$ and all $p, q \in C$ with $q \leq p$ there is a term $t \in S$ and constants $\bar{y} \in C^m$ such that $t(\bar{y}, p) \neq t(\bar{y}, q)$ and $(t(\bar{y}, p), t(\bar{y}, q))(k)$ lies in the monolith of \mathbb{C}_k for each $k \in K$.

Let $\mathbb{B} \leq \prod_{l \in L} \mathbb{C}_l$ be a subdirect representation of \mathbb{B} by subdirectly irreducible algebras, and let $K \subseteq L$ be the coordinates of the machine type SI's. Pick $t \in S$ such that there are constants $\bar{y} \in B^m$ so that $\pi_K(t(\bar{y}, a)) \neq \pi_K(t(\bar{y}, b))$ and $(t(\bar{y}, a), t(\bar{y}, b))(k)$ lies in the monolith of \mathbb{C}_k for each $k \in K$. This is possible by the remarks in the above paragraph and by the hypotheses of the lemma. More precisely, possibilities (1) and (2) mean that a Turing machine action can differentiate between a and b , and possibility (3) means that we should take the term t to

be the identity. Let $c = t(\overline{y}, a)$ and $d = t(\overline{y}, b)$. Since $b \leq a$ and all operations are monotone, $d \leq c$.

If $F \in \mathcal{L} \cup \mathcal{R}$ and $a_1, a_2 \in B$ then either $F(a_1, a_2, c)(k) = 0$ or $F(a_1, a_2, c)(k) = \mathcal{T}(c(k))$. From the structure of the monolith of \mathbb{C}_k as described at the start of this proof, and using the fact that there are only finitely many non-isomorphic machine type SI's, all finite, the conclusion of the lemma follows. \square

Let the sets S and T be as in Lemma 4.21 and define

$$(4.19) \quad \Gamma_{\mathcal{T}}(w, x, y, z) = \bigvee_{t \in S} \exists \overline{n} [w = t(\overline{n}, y) \wedge x = t(\overline{n}, z)].$$

Given Lemmas 4.18 and 4.16, define

$$(4.20) \quad \psi_{\mathcal{T}}(w, x, y, z) = \exists t \left[\bigvee_{G \in T} \exists \overline{b} [w = J'(w, t, G(\overline{b}, y)) \wedge x = J'(w, t, G(\overline{b}, z))] \right].$$

If $c, d \in B$ satisfy the conclusion of Lemma 4.21, then c, d also satisfy the hypotheses of Lemma 4.18. In this case, $(r, s) \in \text{Cg}^{\mathbb{B}}(c, d)$ if and only if $\mathbb{B} \models \psi_{\mathcal{T}}(r, s, c, d)$. Since we will be employing a strategy similar to the proof of Theorems 4.15 and 4.20, where a Maltsev chain is broken into 2 decreasing segments, let

$$(4.21) \quad \psi_4(w, x, y, z) = \exists t [\psi_{\mathcal{T}}(w, t, y, z) \wedge \psi_{\mathcal{T}}(x, t, y, z)].$$

Theorem 4.22. *Let $a, b \in B$ be distinct and such that $b \leq a$ and $e_i(\overline{n}, a) = e_i(\overline{n}, b)$ for all $i \in \{0, 1, 2\}$ and all $\overline{n} \in B^2 \cup B$. If one of the following*

- (1) *there is $F \in \mathcal{L} \cup \mathcal{R}$ and $u, v \in B$ such that $F(u, v, a) \neq F(u, v, b)$, or*
- (2) *there is $F \in \mathcal{L} \cup \mathcal{R}$ and $u, v \in B$ such that $F(u, v, I(a)) \neq F(u, v, I(b))$,*

holds, then the congruence $\text{Cg}^{\mathbb{B}}(a, b)$ has a principal subcongruence witnessed by $\Gamma_{\mathcal{T}}(-, -, a, b)$ and defined by ψ_4 :

$$\mathbb{B} \models \exists c, d [c \neq d \wedge \Gamma_{\mathcal{T}}(c, d, a, b) \wedge \Pi_{\psi_4}(c, d)].$$

Proof. By hypothesis, Lemma 4.21 holds. Let T and S be the finite sets of terms depending only on $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ guaranteed by the conclusion of Lemma 4.21, and let c and d be such that $c = t(\overline{m}, a)$, $d = t(\overline{m}, b)$ for some $t \in S$ and $\overline{m} \in B$, and if

$$F(x) \in \{\text{id}(x)\} \cup \{F_1(a_1, b_1, F_2(a_2, b_2, \dots F_n(a_n, b_n, x) \dots)) \mid \\ n \in \mathbb{N}, F_i \in \mathcal{L} \cup \mathcal{R}, \text{ and } a_i, b_i \in B\},$$

then there is $G \in T$ and $\overline{b} \in B$ such that $F(c) = G(\overline{b}, c)$ and $F(d) = G(\overline{b}, d)$. In either case, $\mathbb{B} \models \Gamma_{\mathcal{T}}(c, d, a, b)$ and c and d satisfy the hypotheses of Lemma 4.18.

Since $a \geq b$ and the operations of \mathbb{B} are monotonic, $c \geq d$. Suppose now that $(r, s) \in \text{Cg}^{\mathbb{B}}(c, d)$. Then using the same argument as in the proof of Theorem 4.15, there are decreasing Maltsev chains $r = r_1, \dots, r_m = t$ and $s = s_1, \dots, s_n = t$ with associated polynomials generated by fundamental translations. Using first Lemma 4.18, and then Lemma 4.16, we have that there are constants ρ and ρ' such that

$$r = J'(r, \rho, G(\overline{b}, c)), \quad t = J'(r, \rho, G(\overline{b}, c)) = J'(s, \rho', G'(\overline{b}', d)), \\ s = J'(s, \rho', G'(\overline{b}', c))$$

for some $G, G' \in T$ and constants $\overline{b}, \overline{b}' \in B$. Hence $\mathbb{B} \models \psi_{\mathcal{T}}(r, t, c, d) \wedge \psi_{\mathcal{T}}(s, t, c, d)$, completing the proof. \square

The last case where $a, b \in B$ differ at a coordinate that is small but does not satisfy $\exists \bar{n}[e_i(\bar{n}, x) \approx x]$ remains. From Lemma 4.17, we know that there are only 3 isomorphism types for such SI's. If the coordinate is isomorphic to $\{0, C\}$, then the lemmas used in the sequential case apply. We are therefore concerned with the remaining two isomorphism types. To this end, let

$$(4.22) \quad \Gamma_I(w, x, y, z) = \exists u, v [u = I(y) \wedge v = I(z) \wedge \Gamma_{(\cdot)}(w, x, u, v)].$$

Lemma 4.23. *Suppose that $a, b \in B$ are distinct, but that $I(x)$ is the only fundamental operation that distinguishes them. Then the congruence $\text{Cg}^{\mathbb{B}}(a, b)$ has a principal subcongruence witnessed by $\Gamma_I(-, -, a, b)$ and defined by ψ_3 (see (4.18) and (4.22)):*

$$\mathbb{B} \models \exists c, d [c \neq d \wedge \Gamma_I(c, d, a, b) \wedge \Pi_{\psi_3}(c, d)].$$

Proof. Let $\mathbb{B} \leq \prod_{l \in L} \mathbb{C}_l$ be a subdirect representation of \mathbb{B} by subdirectly irreducible algebras. If $a, b \in B$ are distinct and only distinguished by $I(x)$, then it must be that $a(l) \neq b(l)$ if and only if $\mathbb{C}_l \cong \mathbb{W}$ or $\mathbb{C}_l \cong \{0, H, M_1^0\}$ (see (4.14) and Lemma 4.17).

In this case, if $a' = I(a)$ and $b' = I(b)$, then a' and b' satisfy the hypotheses of Theorem 4.20 and thus $\text{Cg}^{\mathbb{B}}(a', b')$ has a principal subcongruence witnessed by $\Gamma_{(\cdot)}(-, -, a', b')$ and defined by ψ_3 . Therefore $\text{Cg}^{\mathbb{B}}(a, b)$ has a principal subcongruence witnessed by $\Gamma_I(-, -, a, b)$ and defined by ψ_3 , as claimed. \square

Theorem 4.24. *If \mathcal{T} halts then $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ has definable principal subcongruences.*

Proof. Let

$$\Gamma(w, x, y, z) = \Gamma_1(w, x, y, z) \vee \Gamma_{(\cdot)}(w, x, y, z) \vee \Gamma_{\mathcal{T}}(w, x, y, z) \vee \Gamma_I(w, x, y, z)$$

(see Theorem 4.15 and equations (4.16), (4.19), and (4.22) for definitions of these), and

$$\psi(w, x, y, z) = \psi_2(w, x, y, z) \vee \psi_3(w, x, y, z) \vee \psi_4(w, x, y, z)$$

(see equations (4.13), (4.18), and (4.21) for definitions of these). We claim that $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ has definable principal congruences witnessed by Γ and ψ :

$$\mathcal{V}(\mathbb{A}'(\mathcal{T})) \models \forall a, b [a \neq b \rightarrow \exists c, d [c \neq d \wedge \Gamma(c, d, a, b) \wedge \Pi_{\psi}(c, d)]].$$

Let $\mathbb{B} \in \mathcal{V}(\mathbb{A}'(\mathcal{T}))$ with $a, b \in B$ distinct and let $\mathbb{B} \leq \prod_{l \in L} \mathbb{C}_l$ be a subdirect representation by subdirectly irreducible algebras. Since a and b are distinct, there is some $l \in L$ such that $a(l) \neq b(l)$. Let

$$K = \{l \in L \mid a(l) \neq b(l)\}.$$

For $k \in K$, the case distinction breaks down as follows:

- (1) $\mathbb{C}_k \models e_i(\bar{n}, x) \approx x$ for some $i \in \{0, 1, 2\}$ and some $\bar{n} \in C_k^2 \cup C_k$. In this case, Theorem 4.15 applies.
- (2) The previous case does not apply, but there is some \mathbb{C}_k of sequential type. If this is the case, there is some $u \in B$ such that $u \cdot a \neq u \cdot b$ or $a \cdot u \neq b \cdot u$, or the machine operations $\mathcal{L} \cup \mathcal{R}$ cannot distinguish between a and b . In this case, Theorem 4.20 applies.
- (3) The previous cases do not apply, but there is some \mathbb{C}_k of machine type. If this is the case, there is some machine operation in $\mathcal{L} \cup \mathcal{R}$ that can distinguish between a and b . In this case, Theorem 4.22 applies.

- (4) The previous cases do not apply, so there must be some small \mathbb{C}_k that models $e_i(\bar{y}, x) \approx 0$ for all $i \in \{0, 1, 2\}$ (see Lemma 4.17). If $C_k = \{0, C\}$ or $\mathbb{C} \cong \mathbb{W}$, then Theorem 4.20 applies. If $C_k = \{0, H, M_1^0\}$ then either Theorem 4.20 applies (if $a(k) = M_1^0$) or Lemma 4.23 applies (if $a(k) = H$).

Since the SI's of $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ either satisfy $e_i(\bar{n}, x) \approx x$ for some $i \in \{0, 1, 2\}$ and \bar{n} , are of sequential type, are of machine type, or are isomorphic to one of the small algebras given in Lemma 4.17, this completes the proof. \square

One application of definable principal subcongruences is in defining the subdirectly irreducible members of some class of algebras. If \mathcal{C} is a class of algebras with definable principal subcongruences witnessed by congruence formulas Γ and ψ , then

$$\mathcal{C} \models \forall a, b [a \neq b \rightarrow \exists c, d [c \neq d \wedge \Gamma(c, d, a, b) \wedge \Pi_\psi(c, d)]] ,$$

and the sentence

$$\sigma = \exists r, s [r \neq s \wedge \forall a, b [a \neq b \rightarrow \exists c, d [\Gamma(c, d, a, b) \wedge \psi(r, s, c, d)]]]$$

defines the subdirectly irreducible algebras in \mathcal{C} . Baker and Wang [2] use this to prove the following theorem.

Theorem 4.25 (Baker, Wang [2]). *A variety \mathcal{V} with definable principal subcongruences is finitely based if and only if the class of subdirectly irreducible members of \mathcal{V} is finitely axiomatizable.*

In particular, if $\kappa(\mathcal{V}) < \omega$ then the class of subdirectly irreducible members of \mathcal{V} is finitely axiomatizable since there are only finitely many of them, all finite.

Corollary 4.26. *If \mathcal{T} halts, then $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ is finitely based.*

5. IF \mathcal{T} DOES NOT HALT

In the case where \mathcal{T} halts, every sequentiable subdirectly irreducible algebra is finite and there are only finitely many of them. In the case where \mathcal{T} does not halt, McKenzie [6] and the additions from Section 3 show that the algebra $\mathbb{S}_{\mathbb{Z}}$ (defined in Section 3) is a member of $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$. McKenzie [8] uses $\mathbb{S}_{\mathbb{Z}}$ to show that if \mathcal{T} does not halt, then $\mathbb{A}(\mathcal{T})$ is inherently nonfinitely based. Although $\mathbb{S}_{\mathbb{Z}}$ is not subdirectly irreducible, it contains an infinite subalgebra \mathbb{S}_{ω} which is, and every finite sequentiable SI can be embedded in it. We will use the presence of $\mathbb{S}_{\mathbb{Z}}$ to prove that DPSC is undecidable as well.

An algebra \mathbb{C} is said to be *finitely subdirectly irreducible (FSI)* if for all $a, b, c, d \in C$ such that $a \neq b$ and $c \neq d$, $\text{Cg}^{\mathbb{C}}(a, b) \cap \text{Cg}^{\mathbb{C}}(c, d) \neq \mathbf{0}$ (i.e. $\mathbf{0}$ is meet irreducible). Every SI is FSI, but not every FSI is SI.

Theorem 5.1. *The class of finitely subdirectly irreducible algebras in $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ is not axiomatizable if \mathcal{T} does not halt.*

Proof. We will use an ultrapower argument. Suppose to the contrary that the class of finitely subdirectly irreducible algebras in $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ is axiomatizable, say by Φ . \mathcal{T} does not halt if and only if $\mathbb{S}_{\mathbb{Z}} \in \mathcal{V}(\mathbb{A}'(\mathcal{T}))$. Let \mathbb{S} be an ultrapower of $\mathbb{S}_{\mathbb{Z}}$, so that \mathbb{S} satisfies all first order properties of $\mathbb{S}_{\mathbb{Z}}$. In particular, since $\mathbb{S}_{\mathbb{Z}} \models \Phi$, we have that $\mathbb{S} \models \Phi$, so $\mathbf{0}$ is meet irreducible in $\text{Con}(\mathbb{S})$. We will now give some first-order properties of $\mathbb{S}_{\mathbb{Z}}$ which we will make use of.

Let

$$A = \{\alpha \in S_{\mathbb{Z}} \mid \exists \beta [\alpha \cdot \beta \neq 0]\} \quad \text{and} \quad B = \{\beta \in S_{\mathbb{Z}} \mid \exists \alpha [\alpha \cdot \beta \neq 0]\}.$$

Then in $\mathbb{S}_{\mathbb{Z}}$, for each $\alpha \in A$ there is a unique $\beta \in B$ such that $\alpha \cdot \beta \neq 0$, and for each $\beta \in B$, there is a unique $\alpha \in A$ such that $\alpha \cdot \beta \neq 0$. This gives us that $|A| = |B|$. We also have

$$A \cap B = \emptyset \quad \text{and} \quad S_{\mathbb{Z}} = A \cup B \cup \{0\}.$$

For $b \in B$, let

$$A^n \cdot b = \{\alpha_1 \cdots \alpha_m \cdot b \mid 0 \leq m \leq n \text{ and } \alpha_1, \dots, \alpha_m \in A\}.$$

Then $|A^n \cdot b| = n + 2$. Furthermore, for $b, c \in B$,

$$\text{if } (A^n \cdot b) \cap (A^m \cdot c) \neq \{0\} \quad \text{then} \quad b \in A^m \cdot c \text{ or } c \in A^n \cdot b.$$

Lastly, if $F(x)$ is a fundamental translation in $\mathbb{S}_{\mathbb{Z}}$, then $F(A^n \cdot b) \subseteq A^{n+1} \cdot b$. All of these sets and properties are first-order definable and hold in $\mathbb{S}_{\mathbb{Z}}$, so their analogues hold in \mathbb{S} as well.

We will now begin to examine \mathbb{S} . For $b \in B$, define the *orbit of b* to be

$$b^A = \bigcup_{n \in \mathbb{N}} A^n \cdot b.$$

Since $|A^n \cdot b| = n + 2$, the set b^A is countable. Suppose now that there are $b, c \in B$ such that $b^A \cap c^A = \{0\}$ (this will be the case if the ultrapower is nonprincipal). Then by the properties above, $\text{Cg}^{\mathbb{S}}(b, 0)$ relates the orbit of b to 0 and is the identity relation elsewhere. A similar statement is true of $\text{Cg}^{\mathbb{S}}(c, 0)$. It follows that the two congruences meet to $\mathbf{0}$, which contradicts $\mathbf{0}$ being meet irreducible in $\text{Con}(\mathbb{S})$.

Suppose now that if $b, c \in B$ then $b^A \cap c^A \neq \{0\}$. By the properties above, we have that either $b \in c^A$ or $c \in b^A$. Without loss of generality, assume that $b \in c^A$. There is a finite number n such that $\alpha_1 \cdots \alpha_n \cdot c = b$, so since this is true for all b, c , we have that $\bigcup_{b \in B} b^A$ is countable. Since $B = \bigcup_{b \in B} b^A$, it must be that B is countable. The property that $|A| = |B|$ and $S = A \cup B \cup \{0\}$ therefore gives us that \mathbb{S} is also countable. Thus the ultrapower is principal, and $\mathbb{S} \cong \mathbb{S}_{\mathbb{Z}}$. \square

Corollary 5.2. $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ does not have definable principal subcongruences if \mathcal{T} does not halt.

Proof. Suppose that $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ has definable principal subcongruences witnessed by Γ and ψ , and let

$$\begin{aligned} \zeta = \forall a, b, a', b' [(a \neq b) \wedge (a' \neq b') \rightarrow \exists c, d, c', d' [\Gamma(c, d, a, b) \wedge \Gamma(c', d', a', b') \\ \wedge \exists r, s [r \neq s \wedge \psi(r, s, c, d) \wedge \psi(r, s, c', d')]]]. \end{aligned}$$

For $\mathbb{B} \in \mathcal{V}(\mathbb{A}'(\mathcal{T}))$, we have that $\mathbb{B} \models \zeta$ if and only if \mathbb{B} is finitely subdirectly irreducible. This contradicts Theorem 5.1, so $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ cannot have definable principal subcongruences as we assumed. \square

The above results, in addition to the results from the previous section, yield the following theorem.

Theorem 5.3. *The following are equivalent.*

- (1) \mathcal{T} halts.
- (2) $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ has definable principal subcongruences.

(3) $\mathcal{V}(\mathbb{A}'(\mathcal{T}))$ is finitely based.

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